WORKED SOLUTION

STEP II 2016, Question 6

Pure • Differential Equations and Chebyshev Polynomials

STEP II 2016, Question 6

Read the question before you go on. It is in the STEP Support Programme's question database:

step.maths.org/questions/16-s2-q6

The key idea

Every part is a matter of substituting a proposed function into the differential equation and checking the condition at $x = 1$; the uniqueness assumption then does the rest. The trick in (iii) and (iv) is never to expand y_n (we don't know it explicitly), but to use the equation $(1 - x^2)y_n'^2 = n^2(1 - y_n^2)$ itself to eliminate the derivative.

Solution

Throughout, y_k satisfies

$$(1 - x^2)(y_k'(x))^2 = k^2(1 - y_k(x)^2), \qquad y_k(1) = 1,$$

which is just $(*)$ rearranged. To show a function equals y_k, it is enough (by the assumed uniqueness) to show it satisfies $(*)$ with that value of k and takes the value 1 at $x = 1$.

Part (i)

With $k = 1$ the equation is

$$(1 - x^2)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + y^2 = 1.$$

For $y = x$ we have $y' = 1$, and the left-hand side is $(1 - x^2) + x^2 = 1$, as required. Also $y(1) = 1$, so $y_1(x) = x$.

Part (ii)

With $k = 2$ the equation is

$$(1 - x^2)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 4y^2 = 4.$$

For $y = 2x^2 - 1$ we have $y' = 4x$, so

$$\begin{aligned}(1-x^2)(4x)^2 + 4(2x^2-1)^2 &= 16x^2 - 16x^4 + 16x^4 - 16x^2 + 4\\ &= 4.\end{aligned}$$

Since $y(1) = 2 - 1 = 1$, we get $y_2(x) = 2x^2 - 1$.

Part (iii)

Write $y = y_n(x)$ for short. Differentiating $z = 2y^2 - 1$ by the chain rule gives $z' = 4yy'$, so

$$(1-x^2)z'^2 = 16y^2\,(1-x^2)y'^2 = 16y^2 \cdot n^2(1-y^2),$$

using the equation for y_n. Also

$$4n^2z^2 = 4n^2(2y^2-1)^2 = 4n^2(4y^4 - 4y^2 + 1).$$

Adding,

$$(1-x^2)z'^2 + 4n^2z^2 = 16n^2y^2 - 16n^2y^4 + 16n^2y^4 - 16n^2y^2 + 4n^2 = 4n^2,$$

which is the required equation.

? **Ask yourself** What does this equation have to do with y_{2n}?

It is exactly $(*)$ with $k = 2n$, since $(2n)^2 = 4n^2$. Moreover $z(1) = 2y_n(1)^2 - 1 = 2 - 1 = 1$. So z is a solution of $(*)$ with $k = 2n$ taking the value 1 at $x = 1$, and by uniqueness

$$y_{2n}(x) = 2(y_n(x))^2 - 1.$$

As a check, $n = 1$ gives $y_2 = 2x^2 - 1$, agreeing with (ii), and $n = 2$ gives $y_4 = 2(2x^2-1)^2 - 1 = 8x^4 - 8x^2 + 1$.

Part (iv)

Write $u = y_m(x)$, so $v = y_n(u)$. By the chain rule,

$$v'(x) = y_n'(u)\,y_m'(x).$$

Hence, using first the equation for y_m (at the point x) and then the equation for y_n (at the point u),

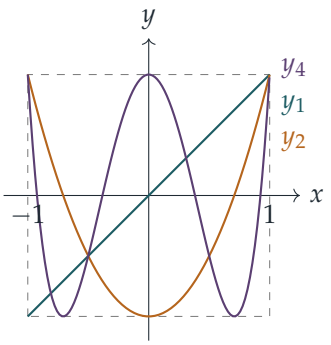
$$\begin{aligned}(1-x^2)v'^2 &= y_n'(u)^2\,(1-x^2)y_m'(x)^2\\ &= y_n'(u)^2 \cdot m^2(1-u^2)\\ &= m^2\,(1-u^2)y_n'(u)^2\\ &= m^2 \cdot n^2(1-y_n(u)^2) = m^2n^2(1-v^2).\end{aligned}$$

So $(1-x^2)v'^2 + (mn)^2v^2 = (mn)^2$, which is $(*)$ with $k = mn$. Finally

$$v(1) = y_n(y_m(1)) = y_n(1) = 1,$$

so by uniqueness $v(x) = y_{mn}(x)$, as required.

The equation for y_n is applied at the point u rather than x: it holds for every value of its variable, so we may use it with u in place of x. This is the step that makes (iv) work.



All three curves pass through $(1,1)$ and stay between -1 and 1 on $[-1,1]$, as the equation forces: $(1-x^2)y'^2 \geq 0$ there, so $k^2y^2 \leq k^2$.

Checks

- Part (iii) with $n = 2$ predicts $y_4(x) = 8x^4 - 8x^2 + 1$. Substituting directly, $y_4' = 32x^3 - 16x$ and $(1-x^2)(32x^3-16x)^2 + 16(8x^4-8x^2+1)^2$ simplifies to 16. At $x = 0.3$, $y_4 = 0.3448$.
- Part (iv) with $m = n = 2$: $y_2(y_2(x)) = 2(2x^2-1)^2 - 1$, which is the same y_4 as from (iii), as it must be.
- A computer check confirmed with exact arithmetic that the polynomials built this way satisfy $(*)$ for every k up to 24, and that $y_n(y_m(x)) = y_{mn}(x)$ for $m, n \leq 4$.

Common slips

- Forgetting the condition at $x = 1$. Satisfying $(*)$ alone isn't enough: $-y_k$ also satisfies it.
- Trying to expand y_n explicitly in (iii) or (iv). Nothing explicit is known about y_n; use its differential equation to replace $(1-x^2)y_n'^2$.
- In (iv), using the equation for y_n with x instead of $u = y_m(x)$, which gives the factor $(1-x^2)$ in the wrong place.

Beyond STEP

The functions y_k are the Chebyshev polynomials T_k: putting $x = \cos\theta$ turns $(*)$ into $(\mathrm{d}y/\mathrm{d}\theta)^2 = k^2(1-y^2)$, solved by $y = \cos k\theta$, so $T_k(\cos\theta) = \cos k\theta$. Part (iii) is then the

double-angle formula and part (iv) says $\cos\big(n(m\theta)\big) = \cos(mn\theta)$. A word of caution: the uniqueness you were allowed to assume is not quite true, since the constant $y = 1$ also satisfies $(*)$ and $y(1) = 1$. Equations like this one, where the derivative is only determined up to sign, are exactly where uniqueness theorems need care; Boyce and DiPrima, *Elementary Differential Equations*, discuss existence and uniqueness of solutions.

Beyond STEP worked solutions: https://beyondstep.keithmcnulty.org/. Solution text CC0 1.0; the question itself is © Cambridge University Press & Assessment and is not reproduced here.