

9.8 Nobody Gets Their Own Shirt

WORKSHEET

The question

Starting with the result $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, prove that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C).$$

Write down, without proof, the corresponding result for four events A, B, C and D .

A pack of n cards, numbered $1, 2, \dots, n$, is shuffled and laid out in a row. The result of the shuffle is that each card is equally likely to be in any position in the row. Let E_i be the event that the card bearing the number i is in the i th position in the row. Write down the following probabilities:

- (i) $P(E_i)$;
- (ii) $P(E_i \cap E_j)$, where $i \neq j$;
- (iii) $P(E_i \cap E_j \cap E_k)$, where $i \neq j, j \neq k$ and $k \neq i$.

Hence show that the probability that at least one card is in the same position as the number it bears is

$$1 - \frac{1}{2!} + \frac{1}{3!} - \dots + (-1)^{n+1} \frac{1}{n!}.$$

Find the probability that exactly one card is in the same position as the number it bears.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

 **Pause and try** Try $n = 3$: list all six orders of cards 1, 2, 3 and count how many have no card in its own place. Then try $n = 4$, if you are patient (there are 24 orders).

STAGE 1 Inclusion–exclusion for three events

 **Ask yourself** How can you write $A \cup B \cup C$ as a union of two events, so that the two-event formula applies?

STAGE 2 The probabilities

 **Ask yourself** If card i is in position i , how many ways are there to arrange the rest? And if cards i and j are both fixed?

STAGE 3 Adding it up

 **Ask yourself** In the inclusion–exclusion formula for $P(E_1 \cup \dots \cup E_n)$, how many terms are there involving intersections of k events, and what is their total?

STAGE 4 Exactly one

 **Ask yourself** For exactly one card to be in place, which card is it, and what must the others do?

Try this next

- 1 **Shirts.** (From STEP III, 2004.) A team of m players put on m numbered shirts at random. Let $C_i = 1$ if player i wears shirt i , and 0 otherwise. Find $E(C_1)$, $\text{Var}(C_1)$ and $\text{Cov}(C_1, C_2)$, and show that $N = C_1 + \dots + C_m$ has $E(N) = \text{Var}(N) = 1$. For $m = 4$, compare $P(N = 0)$ with the Poisson approximation e^{-1} .
- 2 **A recurrence.** Let D_n be the number of derangements of n objects. By considering where card 1 goes, and whether the card in that position goes to position 1, show that $D_n = (n - 1)(D_{n-1} + D_{n-2})$. Check it against $D_3 = 2$, $D_4 = 9$.
- 3 **Exactly k .** Show that the probability that exactly k cards are in their own places is

$\frac{1}{k!}d_{n-k}$, and that it tends to $\frac{e^{-1}}{k!}$.

- 4 Coprime integers.** Use inclusion–exclusion to count the integers from 1 to 30 divisible by none of 2, 3 and 5. How does your answer relate to $30\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right)$?

Further reading

- **Geoffrey Grimmett & Dominic Welsh, *Probability: An Introduction*** (Oxford University Press). A first-year text that covers inclusion–exclusion and the matching problem.
- **William Feller, *An Introduction to Probability Theory and Its Applications, Vol. 1*** (Wiley), Chapter IV. A classic, with many variations on the matching problem.
- **Miklós Bóna, *A Walk Through Combinatorics*** (World Scientific), the chapter on inclusion–exclusion.

The full guided solution, and what the question leads to at university, is in Section 9.8:
https://keithmcnulty.github.io/step_book_mcnulty/ch09/derangements.html