

9.7 Equal Sums of Powers

WORKSHEET

The question

Let $p_0(x) = (1-x)(1-x^2)(1-x^4)$. Show that $(1-x)^3$ is a factor of $p_0(x)$. If $p_1(x) = xp'_0(x)$ show, by considering factors of the polynomials involved, that $p'_0(1) = 0$ and $p'_1(1) = 0$.

By writing $p_0(x)$ in the form

$$p_0(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + c_7x^7,$$

deduce that

$$\begin{aligned} 1 + 2 + 4 + 7 &= 3 + 5 + 6 \\ 1^2 + 2^2 + 4^2 + 7^2 &= 3^2 + 5^2 + 6^2. \end{aligned}$$

Show that we can write the integers $1, 2, \dots, 15$ in some order as $a_1, a_2, \dots, a_{15}$ in such a way that

$$a_1^r + a_2^r + \dots + a_8^r = a_9^r + a_{10}^r + \dots + a_{15}^r$$

for $r = 1, 2, 3$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself If a polynomial $p(x) = \sum c_k x^k$ has $p(1) = 0$, what does that say about the coefficients? What about $p'(1) = 0$?

STAGE 1 A triple root



Ask yourself Factorise $1 - x^2$ and $1 - x^4$. How many factors of $1 - x$ appear?

STAGE 2 Derivatives at 1



Ask yourself If $(1 - x)^3$ divides p_0 , what power of $(1 - x)$ divides p'_0 ? And then $p_1 = xp'_0$ and p'_1 ?

STAGE 3 Reading off the coefficients



Ask yourself What are $p'_0(1)$ and $p'_1(1)$ in terms of the coefficients c_k ?

STAGE 4 One more factor



Pause and try To make the cubes balance too, we need a fourth factor of $(1 - x)$. Which polynomial should replace p_0 ?

Try this next

- Shifting.** (From STEP II, 2025.) Suppose $\sum a_k = \sum b_k$ and $\sum a_k^2 = \sum b_k^2$ for two lists of the same length m . Show that $\sum a_k^3 + \sum (c + b_k)^3 = \sum b_k^3 + \sum (c + a_k)^3$ for every c . Use this, starting from a balanced split of $\{1, \dots, 8\}$, to find a split of $\{1, \dots, 16\}$ in which the cubes balance as well.
- Squares of squares.** (Also from STEP II, 2025.) The sets $\{1, 3, 4, 5, 9, 11\}$ and $\{2, 6, 7, 8, 10\}$ have equal sums and equal sums of squares. Split $\{n^2, (n+1)^2, \dots, (n+11)^2\}$ into two sets with equal sums, for every positive integer n . (*Hint*: add 0 to the second set.)
- Translation invariance.** Show that if two sets of numbers have equal sums of r th powers for $r = 0, 1, \dots, k$, then so do the sets obtained by adding the same number c to every element. Why does this mean that the question's sets $\{1, \dots, 15\}$ could

equally be replaced by any 16 consecutive integers?

- 4 **Fair sweets.** Sweets have values $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{8}$. Compare the totals A gets with the orders $ABABABAB$ and $ABBABAAB$.

Further reading

- **Jean-Paul Allouche & Jeffrey Shallit**, “The ubiquitous Prouhet–Thue–Morse sequence”, in *Sequences and Their Applications* (Springer, 1999). A survey of the sequence’s many appearances; free online.
- **Matt Parker**, *Things to Make and Do in the Fourth Dimension* (Penguin). Includes equal power sums and other number curiosities.
- **Joshua Cooper & Aaron Dutle**, “Greedy Galois games”, *American Mathematical Monthly* 120 (2013). Fair turn-taking and Thue–Morse.

The full guided solution, and what the question leads to at university, is in Section 9.7:
https://keithmcnulty.github.io/step_book_mcnulty/ch09/equal-power-sums.html