

9.6 Strange Dice

WORKSHEET

The question

I have two dice whose faces are all painted different colours. I number the faces of one of them 1, 2, 2, 3, 3, 6 and the other 1, 3, 3, 4, 5, 6. I can now throw a total of 3 in two different ways using the two number 2's on the first die once each. Show that there are seven different ways of throwing a total of 6.

I now renumber the dice (again only using integers in the range 1 to 6) with the results shown in the following table.

Total shown by the two dice	2	3	4	5	6	7	8	9	10	11	12
Different ways of obtaining the total	0	2	1	1	4	3	8	6	5	6	0

Find how I have numbered the dice explaining your reasoning.

[You will only get high marks if the examiner can follow your argument.]

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself If the first die has p_i faces showing i and the second has q_j faces showing j , how many ways are there to throw a total of t ?

STAGE 1 The extremes



Ask yourself There is no way to throw 2, nor 12. What does that say about the 1s and the 6s? There *are* two ways to throw 3. Which die has the 1s?

STAGE 2 Working upwards



Ask yourself Write down the equation for a total of 4, remembering $b_1 = 0$. What does it say in each of the two cases?

STAGE 3 Case (a) fails



Ask yourself In case (a), write down the equation for a total of 7, and use $b_5 + b_6 = 5$.

STAGE 4 Case (b) succeeds



Ask yourself In case (b), use the totals 6 and 11, and the fact that $a_6 b_6 = 0$.

Try this next

- 1 Tetrahedral dice.** (From STEP III, 2003.) Standard four-faced dice are labelled 1, 2, 3, 4. Find all the ways in which two four-faced dice can be labelled with positive integers so that every total has the same probability as for two standard four-faced dice. (*Hint:* $x + x^2 + x^3 + x^4 = x(1 + x)(1 + x^2)$.)
- 2 Checking the table.** Multiply out $(x + x^4 + x^5 + 3x^6)(2x^2 + x^3 + x^4 + 2x^5)$ and check it against the table in the question.
- 3 Three dice.** Are there three six-sided dice, not all standard, labelled with positive integers, whose totals have the same distribution as three standard dice? (*Hint:* the factors of $(x + \dots + x^6)^3$ now include three copies of $1 - x + x^2$.)
- 4 A coin and an octahedron.** An eight-sided die labelled 1, ..., 8 and a two-sided “die” (a coin) labelled 1, 2: can they be relabelled differently with the same totals?

Further reading

- ▶ **Martin Gardner**, *Penrose Tiles to Trapdoor Ciphers* (Mathematical Association of America). Includes the column on Sicherman dice.
- ▶ **Joseph Gallian & David Rusin**, “Cyclotomic polynomials and nonstandard dice”, *Discrete Mathematics* 27 (1979). The general theory, for dice with any number of faces.
- ▶ **Herbert Wilf**, *generatingfunctionology* (free online), Chapter 1.

The full guided solution, and what the question leads to at university, is in Section 9.6:
https://keithmcnulty.github.io/step_book_mcnulty/ch09/strange-dice.html