

9.5 Making Change

WORKSHEET

The question

Show that you can make up 10 pence in eleven ways using 10p, 5p, 2p and 1p coins.

In how many ways can you make up 20 pence using 20p, 10p, 5p, 2p and 1p coins?

[You are reminded that no credit will be given for unexplained answers.]

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself How many ways are there to make n pence using only 2p and 1p coins? Why does the answer depend only on how many 2p coins you use?

STAGE 1 Ten pence



Ask yourself Organise the count by the largest coins. How many 10p coins can you use? Given that, how many 5p coins?

STAGE 2 Building up layer by layer



Ask yourself Let $b(n)$ be the number of ways to make n pence from 5p, 2p and 1p coins. How can you express $b(n)$ using the function a ?

STAGE 3 Twenty pence



Pause and try Notice the structure: each new coin adds a layer, and each layer is a sum of the previous layer at spacings of the new coin's value. Can you see how a computer would carry out this calculation for £1, or £100?

Try this next

- Two kinds of six.** (From STEP II, 2012.) Show that the coefficient of x^{24} in $(1 - x^6)^{-2}(1 - x^3)^{-1}$ is 15, and find the coefficient of x^n in general. Then show that the coefficient of x^{24} in $(1 - x^6)^{-2}(1 - x^3)^{-1}(1 - x)^{-1}$ is 55, and find the coefficients of x^{25} and x^{66} . Describe a coin-counting problem that these coefficients answer.
- Distinct and odd.** Show that $(1 + x)(1 + x^2)(1 + x^3) \cdots$ is the generating function for partitions into distinct parts. Use $1 + x^k = \frac{1 - x^{2k}}{1 - x^k}$ to show that it equals $\frac{1}{(1-x)(1-x^3)(1-x^5)\cdots}$, and deduce Euler's theorem. Check it for $n = 6$.
- A formula.** Show that the number of ways to make n pence using only 1p, 2p and 5p coins is the nearest integer to $\frac{(n+4)^2}{20}$. (Check it first for small n .)
- Order matters.** In how many ways can you make n pence by putting 1p and 2p coins into a slot one at a time, if the order matters? (You have met this sequence before.)

Further reading

- ▶ **George Andrews & Kimmo Eriksson**, *Integer Partitions* (Cambridge University Press). A short, readable introduction for undergraduates.
- ▶ **George Pólya**, “On picture-writing”, *American Mathematical Monthly* 63 (1956). Pólya’s classic explanation of generating functions using the problem of changing a dollar (there are 292 ways).
- ▶ **Robert Kanigel**, *The Man Who Knew Infinity* (Abacus). The life of Ramanujan, and his partnership with Hardy.

The full guided solution, and what the question leads to at university, is in Section 9.5:
https://keithmcnulty.github.io/step_book_mcnulty/ch09/making-change.html