

9.4 Sequences as Functions

WORKSHEET

The question

Given an infinite sequence of numbers $u_0, u_1, u_2, \dots$, we define the *generating function*, f , for the sequence by

$$f(x) = u_0 + u_1x + u_2x^2 + u_3x^3 + \dots$$

Issues of convergence can be ignored in this question.

- (i) Using the binomial series, show that the sequence given by $u_n = n$ has generating function $x(1-x)^{-2}$, and find the sequence that has generating function $x(1-x)^{-3}$.

Hence, or otherwise, find the generating function for the sequence $u_n = n^2$. You should simplify your answer.

- (ii) (a) The sequence $u_0, u_1, u_2, \dots$ is determined by $u_n = ku_{n-1}$ ($n \geq 1$), where k is independent of n , and $u_0 = a$. By summing the identity $u_n x^n \equiv ku_{n-1}x^n$, or otherwise, show that the generating function, f , satisfies

$$f(x) = a + kx f(x).$$

Write down an expression for $f(x)$.

- (b) The sequence $u_0, u_1, u_2, \dots$ is determined by $u_n = u_{n-1} + u_{n-2}$ ($n \geq 2$) and $u_0 = 0, u_1 = 1$. Obtain the generating function.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

Ask yourself What sequence has generating function $\frac{1}{1-x}$? What about $\frac{1}{1-2x}$? What happens to the sequence if you multiply its generating function by x ?

STAGE 1 The natural numbers and the triangle numbers

Ask yourself Expand $(1-x)^{-2}$ by the binomial series. What is the coefficient of x^n ?

Ask yourself How can you write n^2 in terms of n and $\frac{1}{2}n(n+1)$?

STAGE 2 A recurrence becomes an equation

Ask yourself Multiply $u_n = ku_{n-1}$ by x^n and add up over all $n \geq 1$. What is each side, in terms of f ?

STAGE 4 Explaining $1/89$

Pause and try Ignoring convergence (which is fine here: the series converges for $|x| < 0.618$), put $x = \frac{1}{10}$ in f. What do you get?

Try this next

- Two digits at a time.** (From STEP II, 2022.) Using $F_r \leq 2^{r-n}F_n$ for $r \geq n$, show that $\sum_{r \geq 7} F_r/10^r < 2 \times 10^{-6}$, and hence find the first six digits after the decimal point of $1/89$ with justification. Then find a fraction r/s , with r and s positive integers less than 10000, whose decimal expansion starts 0.0001010203050813213455...
- Solving a recurrence.** (From STEP I, 2004.) Let $a_0 = 2$, $a_1 = 7$ and $a_n = 7a_{n-1} - 10a_{n-2}$. Show that the generating function is $\frac{1}{1-2x} + \frac{1}{1-5x}$, and hence that $a_n = 2^n + 5^n$. If $b_0 = 5$, $b_1 = 10$, $b_2 = 40$, $b_3 = 100$ and $b_n = pb_{n-1} + qb_{n-2}$, find p , q and b_n .
- Counting numbers.** Show that $\frac{1}{9801} = 0.00\ 01\ 02\ 03\ 04\ \dots$, with the two-digit blocks counting upwards. Where does the pattern first go wrong, and why?

- 4 Sums of squares.** Explain why $\frac{1}{1-x}f(x)$ is the generating function of the partial sums $u_0 + u_1 + \cdots + u_n$. Use this and the answer to part (i) to find a formula for $1^2 + 2^2 + \cdots + n^2$.

Further reading

- **Herbert Wilf, *generatingfunctionology*** (A K Peters; free online). The classic introduction. Chapter 1 is perfect after this section.
- **Ronald Graham, Donald Knuth & Oren Patashnik, *Concrete Mathematics*** (Addison-Wesley), Chapter 7.
- **Philippe Flajolet & Robert Sedgewick, *Analytic Combinatorics*** (Cambridge University Press; free online). Far beyond first year, but the opening pages show where the subject leads.

The full guided solution, and what the question leads to at university, is in Section 9.4:
https://keithmcnulty.github.io/step_book_mcnulty/ch09/generating-functions.html