

9.3 Colouring Fence Posts

WORKSHEET

The question

I have n fence posts placed in a line and, as part of my spouse's birthday celebrations, I wish to paint them using three different colours red, white and blue in such a way that no adjacent fence posts have the same colours. (This allows the possibility of using fewer than three colours as well as exactly three.) Let r_n be the number of ways (possibly zero) that I can paint them if I paint the first and the last post red and let s_n be the number of ways that I can paint them if I paint the first post red but the last post either of the other two colours. Explain why $r_{n+1} = s_n$ and find $r_n + s_n$. Hence find the value of $r_{n+1} + r_n$ for all $n \geq 1$.

Prove, by induction, that

$$r_n = \frac{2^{n-1} + 2(-1)^{n-1}}{3}.$$

Find the number of ways of painting n fence posts (where $n \geq 3$) placed in a circle using three different colours in such a way that no adjacent fence posts have the same colours.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Pause and try Count the ways to paint 3 posts in a circle, and 4. Does $3 \times 2^{n-1}$ overcount, and by how much?

STAGE 1 Two kinds of colouring



Ask yourself Take a colouring of $n + 1$ posts with the first and last red. Remove the last post. What can you say about the colouring of the first n posts that remains?



Ask yourself What is $r_n + s_n$?

STAGE 2 The formula by induction



Pause and try Where did the formula come from? The recurrence $r_{n+1} + r_n = 2^{n-1}$ is like a differential equation: a “particular solution” $c \cdot 2^n$ plus a “complementary function” $A(-1)^n$ that solves $r_{n+1} + r_n = 0$. Find c and A .

STAGE 3 The circle



Ask yourself In a circle of n posts, fix the colour of post 1, say red. Now cut the circle between post n and post 1, and straighten it into a line. Which of the line colourings correspond to valid circle colourings?

Try this next

- k colours.** Repeat the question with k colours. Show that $r_{n+1} = s_n$ still holds and that $r_{n+1} + r_n = (k-1)^{n-1}$, find r_n , and deduce the chromatic polynomial of the cycle, $(k-1)^n + (-1)^n(k-1)$.
- No two reds together.** How many ways are there to paint n posts in a line red or white, with no two adjacent posts red? (You should find the Fibonacci numbers.)
- A wheel.** Add a central post adjacent to every post of the circle (a “wheel”). How many ways are there to colour it with 4 colours? (*Hint*: the centre uses one colour, and the circle must use the other three.)
- Necklaces.** If two colourings of the circle count as the same when one is a rotation

of the other, how many different colourings are there for $n = 4$? For $n = 5$? (This is the start of *Burnside's lemma*.)

Further reading

- ▶ **Robin Wilson, *Introduction to Graph Theory*** (Pearson). The standard first course; covers colourings and the four colour problem.
- ▶ **Robin Wilson, *Four Colours Suffice*** (Penguin). The story of the four colour theorem, for a general reader.
- ▶ **Herbert Wilf, *generatingfunctionology*** (free online). Recurrences and how to solve them, including the transfer matrix method.

The full guided solution, and what the question leads to at university, is in Section 9.3:
https://keithmcnulty.github.io/step_book_mcnulty/ch09/fence-posts.html