

9.2 Ramsey's Theorem in Miniature

WORKSHEET

The question

Six points A, B, C, D, E and F lie in three dimensional space and are in general positions, that is, no three are collinear and no four lie on a plane. All possible line segments joining pairs of points are drawn and coloured either gold or silver. Prove that there is a triangle whose edges are entirely of one colour. [**Hint:** consider segments radiating from A .]

Give a sketch showing that the result is false for five points in general positions.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

- ?** **Ask yourself** The six points matter only through which pairs are joined by which colour: the geometry is a distraction. How many segments are there, and how many leave each point?

STAGE 1 Five segments, two colours

- ?** **Ask yourself** Five segments leave A , each gold or silver. What can you say for certain about their colours?

STAGE 2 Look at the far ends

- ?** **Ask yourself** Now look at the triangle BCD . What happens if any one of its sides is gold? What if none of them is?

STAGE 3 Five points are not enough

- ?** **Ask yourself** With five points, each point has only four segments. How might you colour them so that each point has exactly two of each colour?

Try this next

- 1 At least two.** Show that in any two-colouring of K_6 there are in fact at least *two* one-colour triangles. (*Hint:* count the “bichromatic angles”: pairs of edges of different colours at a common vertex. A point with g gold and $5 - g$ silver edges is at the apex of $g(5 - g) \leq 6$ such angles. Each triangle that is not one-colour contains exactly two.)
- 2 Three colours.** If the edges of K_{17} are coloured in three colours, show that there is a one-colour triangle. (*Hint:* one point has at least six edges of one colour.)
- 3 Three or four.** Show that $R(3, 4) \leq 10$: in any gold/silver colouring of K_{10} there is a gold triangle or a silver K_4 . (*Hint:* a point has either at least 4 gold edges or at least 6 silver edges.) In fact $R(3, 4) = 9$.
- 4 Monotone subsequences.** Prove the Erdős–Szekeres theorem for $n = 2$: any sequence of 5 distinct numbers contains an increasing or decreasing subsequence of length 3. Find a sequence of 4 with neither.

Further reading

- ▶ **Miklós Bóna**, *A Walk Through Combinatorics* (World Scientific), the chapters on the pigeonhole principle and Ramsey theory.
- ▶ **Ronald Graham & Joel Spencer**, “Ramsey theory”, *Scientific American* (July 1990). A popular introduction by two leading researchers.
- ▶ **Noga Alon & Joel Spencer**, *The Probabilistic Method* (Wiley). The standard book on Erdős's method; the first chapter is accessible.
- ▶ **Paul Hoffman**, *The Man Who Loved Only Numbers* (Fourth Estate). A biography of Erdős.

The full guided solution, and what the question leads to at university, is in Section 9.2:
https://keithmcnulty.github.io/step_book_mcnulty/ch09/ramsey.html