

9.13 Loaded Dice

WORKSHEET

The question

- (i) Prove that if $x > 0$ then $x + x^{-1} \geq 2$.

I have a pair of six-faced dice, each with faces numbered from 1 to 6. The probability of throwing i with the first die is q_i and the probability of throwing j with the second die is r_j ($1 \leq i, j \leq 6$). The two dice are thrown independently and the sum noted. By considering the probabilities of throwing 2, 12 and 7, show the sums $2, 3, \dots, 12$ are not equally likely.

- (ii) The first die described above is thrown twice and the two numbers on the die noted. Is it possible to find values of q_j so that the probability that the numbers are the same is less than $1/36$?

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself How would you prove $x + \frac{1}{x} \geq 2$ for $x > 0$? When is there equality?

STAGE 1 The extreme totals



Ask yourself In how many ways can you throw a total of 2? A total of 12? Write their probabilities in terms of the q_i and r_j .

STAGE 2 The middle total



Ask yourself A total of 7 can be made in six ways. Which two of them involve only the faces 1 and 6? Write their probabilities using the equations above.



Pause and try Look at what made the proof work. Making 2 and 12 as likely as 7 forces the 1s and 6s to be fairly likely, and then the “mixed” outcomes (1, 6) and (6, 1) alone make 7 too likely. Figure 9.13.1 (in the full section online) shows the cells involved.

STAGE 3 Two throws of one die



Ask yourself If one die is thrown twice, what is the probability that the two numbers are the same, in terms of the q_j ? What is it for a fair die?



Ask yourself Can loading make $\sum q_j^2$ smaller than $\frac{1}{6}$? You know that $\sum q_j = 1$.

Try this next

- Cauchy–Schwarz.** Use the Cauchy–Schwarz inequality, $(\sum a_j b_j)^2 \leq \sum a_j^2 \sum b_j^2$, with $a_j = q_j$ and $b_j = 1$, to give a one-line proof of part (ii).
- Two different dice.** If two *different* loaded dice are each thrown once, can the probability that they show the same number be less than $\frac{1}{6}$? Can it be 0?
- Uniform totals with other faces.** Show that two dice *can* give equally likely totals if their faces are numbered differently: try one die numbered 0, 1, 2, 3, 4, 5 and

the other 0, 6, 12, 18, 24, 30. What totals arise? Why does this not contradict the question?

- 4 **Three dice.** Can three loaded six-sided dice, with faces 1 to 6, make all totals from 3 to 18 equally likely? (*Hint:* use the totals 3, 8, 13 and 18, and AM–GM for six numbers. Why does the polynomial proof fail this time?)

Further reading

- **J. Michael Steele, *The Cauchy–Schwarz Master Class*** (Cambridge University Press). A beautiful introduction to inequalities, starting from AM–GM.
- **Geoffrey Grimmett & Dominic Welsh, *Probability: An Introduction*** (Oxford University Press). Generating functions and their use in proofs like the one above.
- **Ian Stewart, *Galois Theory*** (CRC Press), the opening chapters. The most famous impossibility proof of all, for the quintic.

The full guided solution, and what the question leads to at university, is in Section 9.13:
https://keithmcnulty.github.io/step_book_mcnulty/ch09/loaded-dice.html