

# 9.12 Random Subsets

## WORKSHEET

### The question

The set  $S$  is the set of all integers from 1 to  $n$ . The set  $T$  is the set of all distinct subsets of  $S$ , including the empty set  $\emptyset$  and  $S$  itself. Show that  $T$  contains exactly  $2^n$  sets.

The sets  $A_1, A_2, \dots, A_m$ , which are not necessarily distinct, are chosen randomly and independently from  $T$ , and for each  $k$  ( $1 \leq k \leq m$ ), the set  $A_k$  is equally likely to be any of the sets in  $T$ .

- (i) Write down the value of  $P(1 \in A_1)$ .
- (ii) By considering each integer separately, show that  $P(A_1 \cap A_2 = \emptyset) = \left(\frac{3}{4}\right)^n$ .  
Find  $P(A_1 \cap A_2 \cap A_3 = \emptyset)$  and  $P(A_1 \cap A_2 \cap \dots \cap A_m = \emptyset)$ .
- (iii) Find  $P(A_1 \subseteq A_2)$ ,  $P(A_1 \subseteq A_2 \subseteq A_3)$  and  $P(A_1 \subseteq A_2 \subseteq \dots \subseteq A_m)$ .

## Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



**Ask yourself** How can you describe a subset of  $\{1, \dots, n\}$  by a sequence of  $n$  answers to yes/no questions?

### STAGE 2 Disjoint sets



**Ask yourself** For  $A_1 \cap A_2 = \emptyset$ , what must be true of each element  $i$ ? What is the probability of that, for one element? Why can you multiply over the elements?

### STAGE 3 Subsets



**Ask yourself** For  $A_1 \subseteq A_2$ , which of the four possibilities for an element is forbidden?



**Ask yourself** For  $A_1 \subseteq A_2 \subseteq A_3$ , which of the 8 patterns of membership (in or out of each set) are allowed for an element?

## Try this next

- Covering.** Find  $P(A_1 \cup A_2 = S)$  and  $P(A_1 \cup A_2 \cup \dots \cup A_m = S)$ .
- Same size.** Show that  $P(|A_1| = |A_2|) = \binom{2n}{n}/4^n$ . (*Hint:*  $|A_1| = |A_2|$  exactly when  $|A_1| + |S \setminus A_2| = n$ .) Show that this is about  $1/\sqrt{\pi n}$  for large  $n$ .
- Partitions into three.** Show that the number of ordered pairs of disjoint subsets  $(A, B)$  of  $S$  is  $3^n$ , and find the number of unordered pairs  $\{A, B\}$  of disjoint subsets.
- Sperner for small  $n$ .** For  $n = 3$  and  $n = 4$ , find the largest family of subsets of  $\{1, \dots, n\}$  in which no set contains another.

## Further reading

- Miklós Bóna, *A Walk Through Combinatorics* (World Scientific), the chapters on subsets and partially ordered sets.
- Béla Bollobás, *Combinatorics: Set Systems, Hypergraphs, Families of Vectors and Combinatorial Probability* (Cambridge University Press). A short, elegant book on families of subsets, from Sperner's theorem onwards.

- **Peter Cameron**, *Combinatorics: Topics, Techniques, Algorithms* (Cambridge University Press). A first course with a strong emphasis on counting in two ways.

The full guided solution, and what the question leads to at university, is in Section 9.12:  
[https://keithmcnulty.github.io/step\\_book\\_mcnulty/ch09/random-subsets.html](https://keithmcnulty.github.io/step_book_mcnulty/ch09/random-subsets.html)