

9.10 Always Able to Give Change

WORKSHEET

The question

I am selling raffle tickets for £1 per ticket. In the queue for tickets, there are m people each with a single £1 coin and n people each with a single £2 coin. Each person in the queue wants to buy a single raffle ticket and each arrangement of people in the queue is equally likely to occur. Initially, I have no coins and a large supply of tickets. I stop selling tickets if I cannot give the required change.

- (i) In the case $n = 1$ and $m \geq 1$, find the probability that I am able to sell one ticket to each person in the queue.
- (ii) By considering the first three people in the queue, show that the probability that I am able to sell one ticket to each person in the queue in the case $n = 2$ and $m \geq 2$ is $\frac{m-1}{m+1}$.
- (iii) Show that the probability that I am able to sell one ticket to each person in the queue in the case $n = 3$ and $m \geq 3$ is $\frac{m-2}{m+1}$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself Each £1 customer gives you a coin; each £2 customer takes one away (as change). When do you get stuck?

STAGE 2 Two £2 coins



Ask yourself The first person must have a £1 coin. What can happen with the second person, and in each case, what situation are you left in?



Pause and try Try the same approach for $n = 3$. It works, but the cases multiply. Is there a way to count the successful queues all at once?

STAGE 3 Queues as paths



Ask yourself Draw the number of coins you hold against the number of people served. What does a queue look like? What does a successful queue look like?

STAGE 4 The reflection



Ask yourself Take a bad path. Find the first moment it touches -1 , and reflect the part of the path *before* that moment in the line $y = -1$. Where does the new path start, and where does it end?

Try this next

- 1 Three by cases.** Prove part (iii) by conditioning on the first people in the queue, as in part (ii), to see how much work the reflection principle saves.
- 2 Bertrand's ballot.** In an election, A gets a votes and B gets $b < a$. Show that the probability that A is strictly ahead throughout the count is $\frac{a-b}{a+b}$. (*Hint:* the first vote must be for A ; after that, A must never fall behind B by more than $\dots$)
- 3 Brackets.** List the 5 correct arrangements of three pairs of brackets, and the 5 ways to cut a pentagon into triangles. Can you find a bijection between them?
- 4 A £5 note.** A queue contains m people with £1 coins and n with £5 notes, for a ticket costing £1. Explain why the reflection principle no longer gives a simple answer.

(This generalisation is much harder.)

Further reading

- ▶ **William Feller**, *An Introduction to Probability Theory and Its Applications*, Vol. 1 (Wiley), Chapter III. The classic treatment of random walks and the reflection principle.
- ▶ **Richard Stanley**, *Catalan Numbers* (Cambridge University Press). Over two hundred interpretations, with exercises.
- ▶ **Marc Renault**, “Lost (and found) in translation: André’s actual method and its application to the generalized ballot problem”, *American Mathematical Monthly* 115 (2008). The history of the reflection principle.

The full guided solution, and what the question leads to at university, is in Section 9.10:
https://keithmcnulty.github.io/step_book_mcnulty/ch09/ballot.html