

9.1 Lines Cutting the Plane

WORKSHEET

The question

A plane contains n distinct given lines, no two of which are parallel, and no three of which intersect at a point. By first considering the cases $n = 1, 2, 3$ and 4 , provide and justify, by induction or otherwise, a formula for the number of line segments (including the infinite segments).

Prove also that the plane is divided into $\frac{1}{2}(n^2 + n + 2)$ regions (including those extending to infinity).

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

 **Pause and try** Draw the cases $n = 1, 2, 3, 4$ carefully, keeping the lines in general position (no two parallel, no three through a point). Count the segments and the regions in each case. Can you guess the patterns?

STAGE 1 Counting segments

 **Ask yourself** Concentrate on one line. How many of the other lines cross it, and into how many pieces do they cut it?

STAGE 2 What happens when we add a line?

 **Ask yourself** Suppose $n - 1$ lines are drawn, and you add the n th. It is cut into n segments by the others. What does each segment do to the region it passes through?

Try this next

- 1 Circles.** Show that n circles, each pair meeting in two points and no three through a point, divide the plane into $n^2 - n + 2$ regions.
- 2 Parallel lines.** If the n lines include k that are all parallel to each other (and otherwise general position), how many regions are there?
- 3 Cake numbers.** Prove by induction that n planes in general position divide space into $\frac{1}{6}(n^3 + 5n + 6)$ regions.
- 4 A pattern that breaks.** Put n points on a circle and join every pair with a chord, with no three chords through one point inside the circle. For $n = 1, 2, 3, 4, 5$ you get 1, 2, 4, 8, 16 regions. Count the regions for $n = 6$. (*Hint:* use Euler's formula; each set of four points gives one crossing.) Show that the number is $1 + \binom{n}{2} + \binom{n}{4}$.

Further reading

- **Ronald Graham, Donald Knuth & Oren Patashnik, *Concrete Mathematics*** (Addison-Wesley), Chapter 1. Starts with exactly this problem, and the “pizza” and “zig” variants.
- **Miklós Bóna, *A Walk Through Combinatorics*** (World Scientific). A friendly first course,

with chapters on induction, graphs and Euler's formula.

- **Richard Guy**, “**The strong law of small numbers**”, *American Mathematical Monthly* 95 (1988). Many patterns that look obvious and turn out to be false, including the circle problem in the exercises.

The full guided solution, and what the question leads to at university, is in Section 9.1:
https://keithmcnulty.github.io/step_book_mcnulty/ch09/lazy-caterer.html