

8.8 Period Three Implies Chaos

WORKSHEET

The question

Let $f(x) = 7 - 2|x|$. A sequence $u_0, u_1, u_2, \dots$ is defined by $u_0 = a$ and $u_n = f(u_{n-1})$ for $n > 0$.

- (i) (a) Sketch, on the same axes, the graphs with equations $y = f(x)$ and $y = f(f(x))$.
- (b) Find all solutions of the equation $f(f(x)) = x$.
- (c) Find the values of a for which the sequence $u_0, u_1, u_2, \dots$ has period 2.
- (d) Show that, if $a = \frac{28}{5}$, then the sequence $u_2, u_3, u_4, \dots$ has period 2, but neither u_0 nor u_1 is equal to either of u_2 or u_3 .
- (ii) (a) Sketch, on the same axes, the graphs with equations $y = f(x)$ and $y = f(f(f(x)))$.
- (b) Consider the sequence $u_0, u_1, u_2, \dots$ in the cases $a = 1$ and $a = -\frac{7}{9}$. Hence find all the solutions of the equation $f(f(f(x))) = x$.
- (c) Find a value of a such that the sequence $u_3, u_4, u_5, \dots$ has period 3, but where none of u_0, u_1 or u_2 is equal to any of u_3, u_4 or u_5 .

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself What does the graph of $y = 7 - 2|x|$ look like? Where does it cross the line $y = x$? Where does it send the interval $[-7, 7]$?

STAGE 1 The graph of $f \circ f$



Ask yourself As x goes from -7 to 7 , $f(x)$ goes from -7 up to 7 and back down to -7 . What does $f(f(x))$ do?

STAGE 2 Solving $f(f(x)) = x$



Ask yourself Each of the four straight pieces crosses the diagonal $y = x$ exactly once. Two of the crossings you already know. Why? How can you find the other two without expanding nested absolute values?

STAGE 3 Arriving at a cycle late



Ask yourself How could you have found $\frac{28}{5}$ yourself? Work backwards: which numbers does f send to $-\frac{7}{5}$?

STAGE 4 Three-cycles



Pause and try Compute the sequences starting from $a = 1$ and $a = -\frac{7}{9}$.



Ask yourself How many solutions can $f(f(f(x))) = x$ have? Think about the shape of its graph, as in Stage 1.

Try this next

- A 4-cycle.** Use the covering diagram to predict where a 4-cycle of f should be (three steps in I_2 , one in I_1). Then find it exactly. (*Hint:* the cycle has three positive points and one negative point; write down the four linear equations.)
- The subtlety.** In the proof that period 3 implies period n , explain why the point x found cannot be the point 1 (or any point of the original 3-cycle) when $n \geq 4$.

- 3 Period 5 without period 3.** Sketch the graph of a continuous function on $[1, 5]$ with $g(1) = 3$, $g(3) = 4$, $g(4) = 2$, $g(2) = 5$, $g(5) = 1$, joining these points with straight lines. Draw its covering diagram for the intervals $[1, 2]$, $[2, 3]$, $[3, 4]$, $[4, 5]$. Can you find a loop of length 3? (Sharkovskii's theorem says there is no 3-cycle.)
- 4 The tent map.** Check that $x = 14y - 7$ turns f into the tent map. Then show that the tent map doubles y written in binary, flipping all the digits whenever the leading digit is 1, and use this to find its 3-cycles.

Further reading

- **Tien-Yien Li & James Yorke**, “Period three implies chaos”, *American Mathematical Monthly* 82 (1975). Short, and readable with the intermediate value theorem.
- **Robert Devaney**, *A First Course in Chaotic Dynamical Systems* (CRC Press). Written for undergraduates; covers the tent map, the logistic map and Sharkovskii's theorem with full proofs.
- **Keith Burns & Boris Hasselblatt**, “The Sharkovsky theorem: a natural direct proof”, *American Mathematical Monthly* 118 (2011). A modern proof using covering diagrams.

The full guided solution, and what the question leads to at university, is in Section 8.8:
https://keithmcnulty.github.io/step_book_mcnulty/ch08/period-three.html