

8.7 Chaos You Can Solve

WORKSHEET

The question

- (i) The sequence of numbers $u_0, u_1, \dots$ is given by $u_0 = u$ and, for $n \geq 0$,

$$u_{n+1} = 4u_n(1 - u_n). \quad (*)$$

In the case $u = \sin^2 \theta$ for some given angle θ , write down and simplify expressions for u_1 and u_2 in terms of θ . Conjecture an expression for u_n and prove your conjecture.

- (ii) The sequence of numbers $v_0, v_1, \dots$ is given by $v_0 = v$ and, for $n \geq 0$,

$$v_{n+1} = -pv_n^2 + qv_n + r,$$

where p, q and r are given numbers, with $p \neq 0$. Show that a substitution of the form $v_n = \alpha u_n + \beta$, where α and β are suitably chosen, results in the sequence $(*)$ provided that

$$4pr = 8 + 2q - q^2.$$

Hence obtain the sequence satisfying $v_0 = 1$ and, for $n \geq 0$, $v_{n+1} = -v_n^2 + 2v_n + 2$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself Where have you seen the expression $4s(1-s)$ before, with $s = \sin^2 \theta$? Which identity connects $\sin^2 \theta$, $\cos^2 \theta$ and $\sin 2\theta$?

STAGE 1 Guessing the pattern



Ask yourself What happens to the angle at each step? What is your conjecture for u_n ?

STAGE 2 Proving it



Ask yourself Which proof method fits a statement about every term of a sequence defined by a rule from one term to the next?

STAGE 3 Changing variables



Ask yourself Substitute $v_n = \alpha u_n + \beta$ into the rule for v_{n+1} . You need the result to be $u_{n+1} = 4u_n - 4u_n^2$. How many conditions does that give, and how many unknowns do you have?



Pause and try Three equations but only two unknowns: this is why a condition appears. Can you see, before calculating, why a general quadratic map should not always be equivalent to (*)? (Think about what a linear change of variable can and cannot change.)

Try this next

1 Cycles of the logistic map. (From STEP II, 2002.) The sequence x_n satisfies $x_{n+1} = kx_n(1 - x_n)$.

- Prove that, if $0 < k < 4$ and $0 < x_0 < 1$, then $0 < x_n < 1$ for all n .
- Given that $x_0 = x_1 = x_2 = \cdots = a$, with $a \neq 0$ and $a \neq 1$, find k in terms of a .
- Given instead that $x_0 = x_2 = x_4 = \cdots = a$, with $a \neq 0, 1$, show that $ab^3 - b^2 + (1 - a) = 0$, where $b = k(1 - a)$. Given, in addition, that $x_1 \neq a$, find the possible values of k in terms of a . Hence show that a 2-cycle exists only when $k > 3$.

- 2 Squaring minus two.** Show that $v_{n+1} = v_n^2 - 2$ satisfies the condition of part (ii) (write it as $-pv_n^2 + qv_n + r$ with $p = -1$). Find α and β , and check that your answer agrees with $v_n = 2 \cos(2^n \phi)$.
- 3 A 3-cycle.** Use the formula $u_n = \sin^2(2^n \theta)$ to find a 3-cycle of $u_{n+1} = 4u_n(1 - u_n)$. (Hint: try $\theta = \pi/7$, or $\theta = \pi/9$.) How many points have period exactly 3?
- 4 Losing digits.** A computer stores numbers to about 53 binary places. If you iterate $u_{n+1} = 4u_n(1 - u_n)$ on a computer, roughly how many steps before the computed values bear no relation to the true sequence? Is the computed sequence then useless?

Further reading

- **Robert May**, “Simple mathematical models with very complicated dynamics”, *Nature* 261 (1976). A classic, still very readable.
- **James Gleick**, *Chaos: Making a New Science* (Penguin). The story of May, Feigenbaum and the discovery of chaos.
- **Steven Strogatz**, *Nonlinear Dynamics and Chaos* (CRC Press), Chapter 10. One-dimensional maps, the logistic map and Feigenbaum’s constant, at first-year undergraduate level.
- **Ian Stewart**, *Does God Play Dice?* (Penguin). A popular account of chaos, with a good chapter on the logistic map.

The full guided solution, and what the question leads to at university, is in Section 8.7:
https://keithmcnulty.github.io/step_book_mcnulty/ch08/solvable-chaos.html