

8.6 The Quantum Oscillator

WORKSHEET

The question

For $n = 0, 1, 2, \dots$, the functions y_n satisfy the differential equation

$$\frac{d^2 y_n}{dx^2} - \omega^2 x^2 y_n = -(2n + 1)\omega y_n,$$

where ω is a positive constant, and $y_n \rightarrow 0$ and $dy_n/dx \rightarrow 0$ as $x \rightarrow +\infty$ and as $x \rightarrow -\infty$. Verify that these conditions are satisfied, for $n = 0$ and $n = 1$, by

$$y_0(x) = e^{-\lambda x^2} \quad \text{and} \quad y_1(x) = x e^{-\lambda x^2}$$

for some constant λ , to be determined.

Show that

$$\frac{d}{dx} \left(y_m \frac{dy_n}{dx} - y_n \frac{dy_m}{dx} \right) = 2(m - n)\omega y_m y_n,$$

and deduce that, if $m \neq n$,

$$\int_{-\infty}^{\infty} y_m(x) y_n(x) dx = 0.$$

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself Why should a function like $e^{-\lambda x^2}$ be a sensible guess? What happens to $e^{-\lambda x^2}$ and its derivative as $x \rightarrow \pm\infty$?

STAGE 1 The ground state



Ask yourself Differentiate $y_0 = e^{-\lambda x^2}$ twice, substitute into the equation with $n = 0$, and compare coefficients of x^2 and of the constant term.

STAGE 3 An identity



Ask yourself Differentiate $y_m y'_n - y_n y'_m$ by the product rule. Two of the four terms cancel. Then use the differential equation to replace y''_n and y''_m .

STAGE 4 Orthogonality



Pause and try Integrate the identity from $-\infty$ to ∞ . What is the left-hand side?

Try this next

- The next state.** Show that $y_2 = (4\lambda x^2 - 1)e^{-\lambda x^2}$, with $\lambda = \frac{\omega}{2}$, satisfies the equation with $n = 2$. Check directly that $\int_{-\infty}^{\infty} y_0 y_2 dx = 0$, using $\int_{-\infty}^{\infty} e^{-\omega x^2} dx = \sqrt{\pi/\omega}$ and $\int_{-\infty}^{\infty} x^2 e^{-\omega x^2} dx = \frac{1}{2\omega} \sqrt{\pi/\omega}$.
- No other energies.** Try $y = e^{-\lambda x^2}$ with n replaced by a general number ν . Show that it works only if $\nu = 0$. (In fact, decaying solutions exist only when ν is a non-negative integer.)
- Sine waves.** Use the same method to show that $\int_0^\pi \sin mx \sin nx dx = 0$ for $m \neq n$, starting from $y'' = -n^2 y$ with $y(0) = y(\pi) = 0$.
- A ladder.** Show that if y_n solves the equation, then $y_{n+1} = \omega x y_n - y'_n$ solves it with $n + 1$ in place of n . Use this to generate y_1 and y_2 from y_0 . (This is the “raising operator” of quantum mechanics.)

Further reading

- ▶ **David Griffiths**, *Introduction to Quantum Mechanics* (Cambridge University Press). The standard undergraduate physics text; Chapter 2 treats the harmonic oscillator.
- ▶ **Richard Feynman**, *QED: The Strange Theory of Light and Matter* (Princeton University Press). A popular account of quantum mechanics by one of its masters.
- ▶ **Mary Boas**, *Mathematical Methods in the Physical Sciences* (Wiley). Covers Hermite polynomials and Sturm–Liouville theory for students of mathematics and physics.

The full guided solution, and what the question leads to at university, is in Section 8.6:
https://keithmcnulty.github.io/step_book_mcnulty/ch08/quantum-oscillator.html