

8.4 Spiralling onto a Circle

WORKSHEET

The question

- (i) Show that under the changes of variable $x = r \cos \theta$ and $y = r \sin \theta$, where r is a function of θ with $r > 0$, the differential equation

$$(y + x) \frac{dy}{dx} = y - x$$

becomes

$$\frac{dr}{d\theta} + r = 0.$$

Sketch a solution in the x - y plane.

- (ii) Show that the solutions of

$$(y + x - x(x^2 + y^2)) \frac{dy}{dx} = y - x - y(x^2 + y^2)$$

can be written in the form

$$r^2 = \frac{1}{1 + Ae^{2\theta}}$$

and sketch the different forms of solution that arise according to the value of A .

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself Both equations contain the combinations $x \, dx + y \, dy$ and $x \, dy - y \, dx$ in disguise. What are these in polar coordinates?

STAGE 1 Part (i) in polar coordinates



Ask yourself Check that rearrangement by expanding. Now use the two identities above.

STAGE 2 Part (ii)



Pause and try The new equation is the old one plus extra terms involving $x^2 + y^2 = r^2$. Rewrite it in the same way and find $\frac{dr}{d\theta}$.



Ask yourself This is not linear in r , but it becomes linear in $u = r^{-2}$. What equation does u satisfy?

Try this next

- The spiral.** For the solution $r = e^{-\theta}$ of part (i), show that the curve crosses every ray from the origin at the same angle, 45° . (This is why logarithmic spirals are called *equiangular*.)
- Polar identities.** Prove that $x \, dx + y \, dy = r \, dr$ and $x \, dy - y \, dx = r^2 \, d\theta$.
- A system in time.** Consider the system $\dot{r} = r(1 - r^2)$, $\dot{\theta} = 1$. Show that every solution with $r(0) > 0$ approaches the unit circle as $t \rightarrow \infty$. Write the system in terms of x and y .
- Two cycles.** Find a system of the form $\dot{r} = f(r)$, $\dot{\theta} = 1$ with two limit cycles, at $r = 1$ and $r = 2$, one attracting and one repelling.

Further reading

- **Steven Strogatz, *Nonlinear Dynamics and Chaos*** (CRC Press). Chapter 7 is about limit cycles and the Poincaré–Bendixson theorem.
- **Steven Strogatz, *Sync*** (Hyperion). A popular book about rhythms and synchronisation in nature.

- **James Gleick, *Chaos: Making a New Science*** (Penguin). The story of Lorenz and the discovery of chaos.

The full guided solution, and what the question leads to at university, is in Section 8.4:
https://keithmcnulty.github.io/step_book_mcnulty/ch08/limit-cycle.html