

8.3 Harvesting a Population

WORKSHEET

The question

If there are x micrograms of bacteria in a nutrient medium, the population of bacteria will grow at the rate $(2K - x)x$ micrograms per hour. Show that, if $x = K$ when $t = 0$, the population at time t is given by

$$x(t) = K + K \frac{1 - e^{-2Kt}}{1 + e^{-2Kt}}.$$

Sketch, for $t \geq 0$, the graph of x against t . What happens to $x(t)$ as $t \rightarrow \infty$?

Now suppose that the situation is as described in the first paragraph, except that we remove the bacteria from the nutrient medium at a rate L micrograms per hour where $K^2 > L$. We set $\alpha = \sqrt{K^2 - L}$. Write down the new differential equation for x . By considering a new variable $y = x - K + \alpha$, or otherwise, show that, if $x(0) = K$ then $x(t) \rightarrow K + \alpha$ as $t \rightarrow \infty$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

? **Ask yourself** Before solving anything, sketch the growth rate $(2K - x)x$ as a function of x . Where is it zero? Where positive? What does that tell you about how x will behave?

STAGE 1 Solving the logistic equation

? **Ask yourself** Separate the variables: $\int \frac{dx}{(2K - x)x} = \int dt$. Split the left-hand side into partial fractions.

STAGE 2 Harvesting

? **Ask yourself** Complete the square in $(2K - x)x - L$. Where are its zeros?

🔪 **Pause and try** Now substitute $y = x - K + \alpha$. What equation does y satisfy? Compare it with the original logistic equation.

Try this next

- 1 Starting low.** With harvesting and $K^2 > L$, show that if $x(0) < K - \alpha$ then the population reaches zero in finite time. (*Hint*: what does y do when $y(0) < 0$?)
- 2 Over-harvesting.** If $L > K^2$, write $\beta = \sqrt{L - K^2}$. Solve $x' = -(x - K)^2 - \beta^2$ with $x(0) = 2K$ and find when the population reaches zero.
- 3 Proportional harvesting.** Suppose instead that a fixed *fraction* of the population is removed: $x' = (2K - x)x - hx$. Find the equilibria and show that the population survives for all $h < 2K$. Why is this policy safer?
- 4 Half-time.** In the unharvested case, find the time at which the population reaches $\frac{3}{2}K$.

Further reading

- **Steven Strogatz, *Nonlinear Dynamics and Chaos*** (CRC Press). Chapter 3 is about bifurcations, with this harvesting model as an exercise.
- **Nicholas Britton, *Essential Mathematical Biology*** (Springer). Population models for

undergraduates.

- **Mark Kurlansky, *Cod: A Biography of the Fish That Changed the World*** (Penguin). The history of the cod fishery and its collapse.

The full guided solution, and what the question leads to at university, is in Section 8.3:
https://keithmcnulty.github.io/step_book_mcnulty/ch08/harvesting.html