

8.2 Bounding Without Solving

WORKSHEET

The question

- (i) Let $y(x)$ be a solution of the differential equation $\frac{d^2y}{dx^2} + y^3 = 0$ with $y = 1$ and $\frac{dy}{dx} = 0$ at $x = 0$, and let

$$E(x) = \left(\frac{dy}{dx}\right)^2 + \frac{1}{2}y^4.$$

Show by differentiation that E is constant and deduce that $|y(x)| \leq 1$ for all x .

- (ii) Let $v(x)$ be a solution of the differential equation $\frac{d^2v}{dx^2} + x\frac{dv}{dx} + \sinh v = 0$ with $v = \ln 3$ and $\frac{dv}{dx} = 0$ at $x = 0$, and let

$$E(x) = \left(\frac{dv}{dx}\right)^2 + 2\cosh v.$$

Show that $\frac{dE}{dx} \leq 0$ for $x \geq 0$ and deduce that $\cosh v(x) \leq \frac{5}{3}$ for $x \geq 0$.

- (iii) Let $w(x)$ be a solution of the differential equation

$$\frac{d^2w}{dx^2} + (5\cosh x - 4\sinh x - 3)\frac{dw}{dx} + (w\cosh w + 2\sinh w) = 0$$

with $\frac{dw}{dx} = \frac{1}{\sqrt{2}}$ and $w = 0$ at $x = 0$. Show that $\cosh w(x) \leq \frac{5}{4}$ for $x \geq 0$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

- Ask yourself** For part (i), think of y as the position of a particle and x as time. Then $y'' = -y^3$ is Newton's law with force $-y^3$. What are the kinetic and potential energies?

STAGE 2 Dissipation

- Ask yourself** Differentiate $E = (v')^2 + 2 \cosh v$. Use the differential equation to replace v'' . What sign does the result have?

STAGE 3 Inventing the energy

- Ask yourself** If $E = (w')^2 + F(w)$, then $E' = 2w'w'' + F'(w)w'$. What should $F'(w)$ be, so that the equation can be used to replace w'' and leave only a multiple of $(w')^2$?

- Pause and try** Show that $5 \cosh x - 4 \sinh x - 3 \geq 0$ for all x . (Write it in terms of e^x and look for a perfect square.) Then finish the argument.

Try this next

- Simple harmonic motion.** For $y'' + y = 0$ with $y(0) = A$, $y'(0) = 0$, find the conserved energy and deduce $|y| \leq A$. Check against the exact solution.
- The pendulum.** The angle θ of a pendulum satisfies $\theta'' + \sin \theta = 0$ (in suitable units). Show that $(\theta')^2 - 2 \cos \theta$ is constant. If the pendulum starts at rest at angle θ_0 , show that it never exceeds θ_0 in absolute value.
- Damping.** For $y'' + 2ky' + y = 0$ with $k > 0$, show that $(y')^2 + y^2$ decreases. Does it necessarily tend to zero?
- An escape.** For $y'' - y^3 = 0$ with $y(0) = 1$, $y'(0) = 0$, find a conserved quantity and show that y is not bounded by it. What goes wrong?

Further reading

- **Steven Strogatz, *Nonlinear Dynamics and Chaos*** (CRC Press). Chapters 6 and 7 cover phase planes, conservative systems and Lyapunov functions.

- **D. W. Jordan & P. Smith, *Nonlinear Ordinary Differential Equations*** (Oxford University Press). A standard undergraduate text with many examples of energy methods.
- **Keith Devlin, *The Millennium Problems*** (Basic Books). Includes an accessible account of the Navier–Stokes problem.

The full guided solution, and what the question leads to at university, is in Section 8.2:
https://keithmcnulty.github.io/step_book_mcnulty/ch08/bounding-without-solving.html