

8.1 When Solutions Aren't Unique

WORKSHEET

The question

Find the two solutions of the differential equation

$$\left(\frac{dy}{dx}\right)^2 = 4y$$

which pass through the point (a, b^2) , where $b \neq 0$.

Find two distinct points $(a_1, 1)$ and $(a_2, 1)$ such that one of the solutions through each of them also passes through the origin. Show that the graphs of these two solutions coincide and sketch their common graph, together with the other solutions through $(a_1, 1)$ and $(a_2, 1)$.

Now sketch sufficient members of the family of solutions (for varying a and b) to indicate the general behaviour. Use your sketch to identify a common tangent, and comment briefly on its relevance to the differential equation.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself Take the square root of both sides: $y' = \pm 2\sqrt{y}$. Try $y = (x - c)^2$. Does it satisfy the equation? Is there an even simpler solution?

STAGE 1 Solutions through a point



Ask yourself Which parabolas $y = (x - c)^2$ pass through (a, b^2) ?

STAGE 3 The whole family, and a common tangent



Pause and try Sketch several of the parabolas $y = (x - c)^2$ for different c . Which line touches every one of them? Is that line itself a solution?

STAGE 4 Why it matters



Ask yourself Start at the origin. Which solutions pass through it? Can you build a solution that follows the x -axis for a while and then leaves it?

Try this next

- Two rates.** (From STEP II, 2018.) Using $v = \sqrt{y}$, solve $\frac{dy}{dt} = \alpha y^{1/2} - \beta y$ (with $\alpha, \beta > 0$) and find the non-constant solution y_1 with $y_1(0) = 0$. Do the same for $\frac{dy}{dt} = \alpha y^{2/3} - \beta y$, using $v = y^{1/3}$. Why does $y = 0$ also satisfy both equations with $y(0) = 0$?
- The leaking bucket.** Solve $h' = -k\sqrt{h}$ with $h(0) = h_0 > 0$. When is the bucket empty? Write down two different solutions with $h(T) = 0$ for some T and $h(t) = 0$ for all $t \geq T$.
- Uniqueness restored.** Show that $y' = 2y$ with $y(0) = 0$ has only the solution $y = 0$. (Hint: consider $\frac{d}{dx}(ye^{-2x})$.) What property does $2y$ have that $2\sqrt{y}$ lacks?
- Cubes.** Find all solutions of $y' = 3y^{2/3}$ through the origin.

Further reading

- ▶ **Morris Hirsch, Stephen Smale & Robert Devaney, *Differential Equations, Dynamical Systems, and an Introduction to Chaos*** (Academic Press). Existence and uniqueness, and much of the rest of this chapter.
- ▶ **George Simmons, *Differential Equations with Applications and Historical Notes*** (CRC Press). A classic, full of history; discusses singular solutions and envelopes.
- ▶ **John Norton, “The dome: an unexpectedly simple failure of determinism”, *Philosophy of Science* 75** (2008). Short and readable.

The full guided solution, and what the question leads to at university, is in Section 8.1:
https://keithmcnulty.github.io/step_book_mcnulty/ch08/non-uniqueness.html