

7.9 Only Five Platonic Solids

WORKSHEET

The question

A polyhedron is a solid bounded by F plane faces, which meet in E edges and V vertices. You may assume *Euler's formula*, that $V - E + F = 2$.

In a regular polyhedron the faces are equal regular m -sided polygons, n of which meet at each vertex. Show that

$$F = \frac{4n}{h},$$

where $h = 4 - (n - 2)(m - 2)$.

By considering the possible values of h , or otherwise, prove that there are only five regular polyhedra, and find V , E and F for each.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself Check Euler's formula on a cube: how many vertices, edges and faces does it have? And on a square-based pyramid?



Ask yourself In a regular polyhedron, each face has m edges and each edge borders two faces. What equation does that give between E and F ? Similarly, between E and V ?

STAGE 2 Which values of m and n are possible?



Ask yourself What is the smallest possible value of m (sides of a face) and of n (faces at a vertex)? And since $F > 0$, what sign must h have?



Pause and try For each of the five possibilities, compute h , then F , then $E = \frac{mF}{2}$ and $V = \frac{mF}{n}$. Identify each solid.

Try this next

- 1 Duality.** Show that swapping m and n swaps V and F and leaves E unchanged. Which pairs of Platonic solids are dual to each other? Which is self-dual?
- 2 A football.** A traditional football is made of 12 pentagons and 20 hexagons, with three faces meeting at each vertex. Use double counting and Euler's formula to check that $V - E + F = 2$. Show more generally that any such polyhedron made of pentagons and hexagons, three at each vertex, has exactly 12 pentagons.
- 3 A torus.** Make a "picture frame" polyhedron (a square block with a square hole) and count V , E and F . Check that $V - E + F = 0$.
- 4 Angles at a vertex.** The interior angle of a regular m -gon is $180^\circ(1 - \frac{2}{m})$. Show that n of them fit round a vertex of a convex solid (total less than 360°) exactly when $(m - 2)(n - 2) < 4$, giving a second proof of the classification.

Further reading

- **David Richeson, *Euler's Gem*** (Princeton University Press). The history of $V - E + F = 2$ and how it led to topology. Excellent for a sixth-former.
- **Imre Lakatos, *Proofs and Refutations*** (Cambridge University Press). A dialogue about

Euler's formula and the nature of proof.

- **H. S. M. Coxeter**, *Regular Polytopes* (Dover). The classic book on regular polyhedra and their higher-dimensional cousins.

The full guided solution, and what the question leads to at university, is in Section 7.9:
https://keithmcnulty.github.io/step_book_mcnulty/ch07/platonic-solids.html