

7.8 Four Squares

WORKSHEET

The question

Let a, b, c, d, p, q, r and s be real numbers. By considering the determinant of the matrix product

$$\begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{pmatrix} \begin{pmatrix} z_3 & z_4 \\ -z_4^* & z_3^* \end{pmatrix},$$

where z_1, z_2, z_3 and z_4 are suitably chosen complex numbers, find expressions L_1, L_2, L_3 and L_4 , each of which is linear in a, b, c and d and also linear in p, q, r and s , such that

$$(a^2 + b^2 + c^2 + d^2)(p^2 + q^2 + r^2 + s^2) = L_1^2 + L_2^2 + L_3^2 + L_4^2.$$

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

Ask yourself Recall the two-square version: $(a^2 + b^2)(p^2 + q^2)$ is a sum of two squares because $|z|^2|w|^2 = |zw|^2$ for complex numbers $z = a + bi$, $w = p + qi$. How could matrices extend this to four squares?

STAGE 1 The determinant is a sum of four squares

Ask yourself What is the determinant of $\begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{pmatrix}$? If $z_1 = a + bi$ and $z_2 = c + di$, what is it in terms of a, b, c, d ?

STAGE 3 Reading off the four squares

Pause and try Expand $w_1 = (a + bi)(p + qi) - (c + di)(r - si)$ and $w_2 = (a + bi)(r + si) + (c + di)(p - qi)$ into real and imaginary parts.

Try this next

- A worked example.** Use the formulas for $L_1, \dots, L_4$ with $(a, b, c, d) = (p, q, r, s) = (2, 1, 1, 1)$ to write $49 = 7 \times 7$ as a sum of four squares, and check your answer. Is it the same as the obvious $49 = 7^2 + 0^2 + 0^2 + 0^2$?
- Two squares inside four.** Show that if $c = d = r = s = 0$, the identity reduces to the two-square identity of Section 1.6.
- Numbers needing four squares.** Show that no number of the form $8m + 7$ is a sum of three squares, by considering squares modulo 8.
- Jacobi's formula.** Count the representations of 2 as a sum of four squares (with order and signs mattering), and check that the answer is $8 \times (1 + 2) = 24$.

Further reading

- **Harold Davenport, *The Higher Arithmetic*** (Cambridge University Press). Chapter V proves Lagrange's four-square theorem.
- **John Conway & Derek Smith, *On Quaternions and Octonions*** (A. K. Peters). The arithmetic of quaternions, including a proof of the four-square theorem via quaternion integers.

- **Aigner & Ziegler, *Proofs from THE BOOK***, chapter “Representing numbers as sums of two squares”. The two-square companion to this section.

The full guided solution, and what the question leads to at university, is in Section 7.8:
https://keithmcnulty.github.io/step_book_mcnulty/ch07/four-squares.html