

7.7 Quantities That Never Change

WORKSHEET

The question

In this question, $\mathbf{M}$ and $\mathbf{N}$ are non-singular 2×2 matrices.

The *trace* of the matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is defined as $\text{tr}(\mathbf{M}) = a + d$.

- (i) Prove that, for any two matrices $\mathbf{M}$ and $\mathbf{N}$, $\text{tr}(\mathbf{MN}) = \text{tr}(\mathbf{NM})$ and derive an expression for $\text{tr}(\mathbf{M} + \mathbf{N})$ in terms of $\text{tr}(\mathbf{M})$ and $\text{tr}(\mathbf{N})$.

The entries in matrix $\mathbf{M}$ are functions of t and $\frac{d\mathbf{M}}{dt}$ denotes the matrix whose entries are the derivatives of the corresponding entries in $\mathbf{M}$.

- (ii) Show that

$$\frac{1}{\det \mathbf{M}} \frac{d}{dt} (\det \mathbf{M}) = \text{tr} \left(\mathbf{M}^{-1} \frac{d\mathbf{M}}{dt} \right).$$

- (iii) In this part, matrix $\mathbf{M}$ satisfies the differential equation

$$\frac{d\mathbf{M}}{dt} = \mathbf{MN} - \mathbf{NM},$$

where the entries in matrix $\mathbf{N}$ are also functions of t .

Show that $\det \mathbf{M}$, $\text{tr}(\mathbf{M})$ and $\text{tr}(\mathbf{M}^2)$ are independent of t .

In the case $\mathbf{N} = \begin{pmatrix} t & t \\ 0 & t \end{pmatrix}$, and given that $\mathbf{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ when $t = 0$, find $\mathbf{M}$ as a function of t .

- (iv) In this part, matrix $\mathbf{M}$ satisfies the differential equation

$$\frac{d\mathbf{M}}{dt} = \mathbf{MN},$$

where the entries in matrix $\mathbf{N}$ are again functions of t .

The trace of $\mathbf{M}$ is non-zero and independent of t . Is it necessarily true that $\text{tr}(\mathbf{N}) = 0$?

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself The expression $\mathbf{M}\mathbf{N} - \mathbf{N}\mathbf{M}$ is called the *commutator* of $\mathbf{M}$ and $\mathbf{N}$. When is it zero? What is its trace, given part (i)?

STAGE 2 The derivative of a determinant



Ask yourself Differentiate $\det \mathbf{M} = ad - bc$ by the product rule. Then compute $\mathbf{M}^{-1} \frac{d\mathbf{M}}{dt}$ and its trace, using $\mathbf{M}^{-1} = \frac{1}{\det \mathbf{M}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

STAGE 3 Three conserved quantities



Ask yourself Take the trace of $\dot{\mathbf{M}} = \mathbf{M}\mathbf{N} - \mathbf{N}\mathbf{M}$. For the determinant, use part (ii): what is $\mathbf{M}^{-1}(\mathbf{M}\mathbf{N} - \mathbf{N}\mathbf{M})$?

STAGE 4 An explicit solution



Pause and try With $\mathbf{N} = t\mathbf{I} + t\mathbf{E}$, where $\mathbf{E} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, compute $\mathbf{M}\mathbf{N} - \mathbf{N}\mathbf{M}$ and write down the differential equations for a, b, c, d . Solve them in the order c , then a and d , then b .

STAGE 5 A different equation



Ask yourself If $\dot{\mathbf{M}} = \mathbf{M}\mathbf{N}$, then $\mathbf{N} = \mathbf{M}^{-1}\dot{\mathbf{M}}$, and by part (ii) $\text{tr } \mathbf{N}$ is the logarithmic derivative of $\det \mathbf{M}$. Can the trace of $\mathbf{M}$ stay constant while its determinant changes?

Try this next

- 1 Check the solution.** Verify by differentiating that the matrix $\mathbf{M}(t)$ in Stage 4 satisfies $\dot{\mathbf{M}} = \mathbf{M}\mathbf{N} - \mathbf{N}\mathbf{M}$.
- 2 Similarity.** Show that if $\mathbf{M}(t) = \mathbf{S}^{-1}\mathbf{M}_0\mathbf{S}$, where $\mathbf{M}_0$ is constant and $\mathbf{S}(t)$ satisfies $\dot{\mathbf{S}} = \mathbf{S}\mathbf{N}$, then $\dot{\mathbf{M}} = \mathbf{M}\mathbf{N} - \mathbf{N}\mathbf{M}$. (Hint: $\frac{d}{dt}\mathbf{S}^{-1} = -\mathbf{S}^{-1}\dot{\mathbf{S}}\mathbf{S}^{-1}$.)

3 Volume growth. Let $\mathbf{M}(t) = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{pmatrix}$. Find $\mathbf{N}$ with $\dot{\mathbf{M}} = \mathbf{M}\mathbf{N}$, and check that $\frac{d}{dt} \ln \det \mathbf{M} = \text{tr } \mathbf{N}$.

4 Higher powers. Show directly that $\frac{d}{dt} \text{tr}(\mathbf{M}^k) = 0$ for every positive integer k when $\dot{\mathbf{M}} = \mathbf{M}\mathbf{N} - \mathbf{N}\mathbf{M}$.

Further reading

- **P. G. Drazin & R. S. Johnson, *Solitons: An Introduction*** (Cambridge University Press). The standard undergraduate introduction, including Lax pairs.
- **Thierry Dauxois & Michel Peyrard, *Physics of Solitons*** (Cambridge University Press). Includes the story of Scott Russell's wave.
- **Dwight Neuenschwander, *Emmy Noether's Wonderful Theorem*** (Johns Hopkins University Press). How symmetries give conservation laws.

The full guided solution, and what the question leads to at university, is in Section 7.7:
https://keithmcnulty.github.io/step_book_mcnulty/ch07/lax-pairs.html