

# 7.6 Hyperbolic Rotations and Relativity

## WORKSHEET

### The question

The transformation  $T$  from  $\begin{pmatrix} x \\ y \end{pmatrix}$  to  $\begin{pmatrix} x' \\ y' \end{pmatrix}$  in two-dimensional space is given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where  $u$  is a positive real constant. Show that the curve with equation  $x^2 - y^2 = 1$  is transformed into itself. Find the equations of two straight lines through the origin which transform into themselves.

A line, not necessarily through the origin, which has gradient  $\tanh v$  transforms under  $T$  into a line with gradient  $\tanh v'$ . Show that  $v' = v + u$ .

The lines  $\ell_1$  and  $\ell_2$  with gradients  $\tanh v_1$  and  $\tanh v_2$  transform under  $T$  into lines with gradients  $\tanh v'_1$  and  $\tanh v'_2$  respectively. Find the relation satisfied by  $v_1$  and  $v_2$  that is necessary and sufficient for  $\ell_1$  and  $\ell_2$  to intersect at the same angle as their transforms.

In the case when  $\ell_1$  and  $\ell_2$  meet at the origin, illustrate in a diagram the relation between  $\ell_1$ ,  $\ell_2$  and their transforms.

## Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

**Ask yourself** What identity do  $\cosh u$  and  $\sinh u$  satisfy, analogous to  $\cos^2 u + \sin^2 u = 1$ ? And what are the addition formulas for  $\cosh(u + v)$  and  $\sinh(u + v)$ ?

### STAGE 1 The hyperbola is invariant

**Ask yourself** Which lines through the origin are mapped to themselves? (Look for eigenvectors.)

### STAGE 3 When are angles preserved?

**Ask yourself** The angle  $\psi$  between two lines with gradients  $m_1, m_2$  satisfies  $\tan \psi = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$ . Simplify this when  $m_i = \tanh v_i$ . (*Hint*: put everything over  $\cosh v_1 \cosh v_2$ .)

**Pause and try** When is  $\cosh(v_1 + v_2) = \cosh(v_1 + v_2 + 2u)$ ?

## Try this next

- 1 Composing boosts.** Show that  $T(u)T(w) = T(u + w)$ , where  $T(u)$  is the matrix in the question. What is  $T(u)^{-1}$ ?
- 2 Time dilation.** A spaceship travels at 0.8 times the speed of light. Find its rapidity  $u$  and  $\cosh u$ . If 10 years pass on Earth, how much time passes on the ship? (Use  $t_{\text{ship}} = t_{\text{Earth}} / \cosh u$ .)
- 3 Hyperbolic area.** Show that the area enclosed by the hyperbola  $x^2 - y^2 = 1$ , the  $x$ -axis and the line from the origin to  $(\cosh v, \sinh v)$  is  $\frac{v}{2}$ , just as the corresponding area for a circle is  $\frac{\theta}{2}$ . Why must  $T$  preserve such areas?
- 4 Squeeze.** Change coordinates to  $p = x + y$ ,  $q = x - y$ . Show that  $T$  becomes  $p \mapsto e^u p$ ,  $q \mapsto e^{-u} q$ , and that the hyperbola becomes  $pq = 1$ . Describe the transformation geometrically in these coordinates.

## Further reading

- **Edwin Taylor & John Wheeler, *Spacetime Physics*** (W. H. Freeman). Special relativity

with rapidity and spacetime diagrams, for students with A-level mathematics.

- ▶ **Tevian Dray, *The Geometry of Special Relativity*** (CRC Press). Relativity as hyperbolic trigonometry, in exactly the spirit of this section.
- ▶ **Albert Einstein, *Relativity: The Special and the General Theory*** (various editions). Einstein's own popular account, still very readable.

The full guided solution, and what the question leads to at university, is in Section 7.6:  
[https://keithmcnulty.github.io/step\\_book\\_mcnulty/ch07/hyperbolic-rotations.html](https://keithmcnulty.github.io/step_book_mcnulty/ch07/hyperbolic-rotations.html)