

7.5 Rotating in Three Dimensions

WORKSHEET

The question

Let $\mathbf{n}$ be a vector of unit length in three dimensions. For each vector $\mathbf{r}$, $f(\mathbf{r})$ is defined by $f(\mathbf{r}) = \mathbf{n} \times \mathbf{r}$.

- (i) Given that

$$\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

show that the x -component of $f(f(\mathbf{r}))$ is $-x(b^2 + c^2) + aby + acz$. Show further that

$$f(f(\mathbf{r})) = (\mathbf{n} \cdot \mathbf{r})\mathbf{n} - \mathbf{r}.$$

Explain, by means of a diagram, how $f(f(\mathbf{r}))$ is related to $\mathbf{n}$ and $\mathbf{r}$.

- (ii) Let R be the point with position vector $\mathbf{r}$ and P be the point with position vector $g(\mathbf{r})$, where g is defined by

$$g(\mathbf{s}) = \mathbf{s} + \sin \theta f(\mathbf{s}) + (1 - \cos \theta) f(f(\mathbf{s})).$$

By considering $g(\mathbf{n})$ and $g(\mathbf{r})$ when $\mathbf{r}$ is perpendicular to $\mathbf{n}$, state, with justification, the geometric transformation which maps R onto P .

- (iii) Let R be the point with position vector $\mathbf{r}$ and Q be the point with position vector $h(\mathbf{r})$, where h is defined by

$$h(\mathbf{s}) = -\mathbf{s} - 2f(f(\mathbf{s})).$$

State, with justification, the geometric transformation which maps R onto Q .

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself What does $\mathbf{n} \times \mathbf{r}$ look like geometrically, when $\mathbf{r}$ is perpendicular to $\mathbf{n}$? And when $\mathbf{r}$ is parallel to $\mathbf{n}$?

STAGE 1 Crossing twice



Ask yourself Write $\mathbf{r} = \mathbf{r}_{\parallel} + \mathbf{r}_{\perp}$, where $\mathbf{r}_{\parallel} = (\mathbf{n} \cdot \mathbf{r})\mathbf{n}$ is the component along $\mathbf{n}$. What is $f(f(\mathbf{r}))$ in terms of these?

STAGE 2 The rotation



Ask yourself Compute $g(\mathbf{n})$. Then compute $g(\mathbf{r})$ for $\mathbf{r} \perp \mathbf{n}$, using $f(f(\mathbf{r})) = -\mathbf{r}$ for such $\mathbf{r}$.

STAGE 3 The reflection



Pause and try Simplify $h(\mathbf{s}) = -\mathbf{s} - 2f(f(\mathbf{s}))$ using part (i). What does it do to vectors along $\mathbf{n}$, and to vectors perpendicular to $\mathbf{n}$?

Try this next

- 1 Rotations in the plane.** (From STEP III, 2017.) Show that rotation by θ about the point a in the complex plane sends z to $e^{i\theta}z + a(1 - e^{i\theta})$. Show that rotating by θ about a and then by ϕ about b is a rotation by $\theta + \phi$ about some point c , provided $\theta + \phi$ is not a multiple of 2π . What is it when $\theta + \phi = 2\pi$?
- 2 A specific rotation.** Find the matrix of the rotation by 120° about the axis in the direction $(1, 1, 1)$. Show that it permutes the coordinate axes cyclically.
- 3 Powers of $\mathbf{K}$.** Show that $\mathbf{K}^2\mathbf{r} = (\mathbf{n} \cdot \mathbf{r})\mathbf{n} - \mathbf{r}$ and $\mathbf{K}^3 = -\mathbf{K}$, and deduce Rodrigues' formula from the exponential series.
- 4 Two reflections make a rotation.** Show that reflecting in two planes through the origin, at angle α to each other, is a rotation by 2α about their line of intersection.

Further reading

- **Simon Altmann, *Rotations, Quaternions, and Double Groups*** (Dover). The definitive account of three-dimensional rotations, including Rodrigues' contribution.

- ▶ **Simon Altmann**, “Hamilton, Rodrigues, and the quaternion scandal”, *Mathematics Magazine* 62 (1989). A short, lively history.
- ▶ **John Stillwell**, *Naive Lie Theory* (Springer). Rotations, quaternions and Lie groups for undergraduates.

The full guided solution, and what the question leads to at university, is in Section 7.5:
https://keithmcnulty.github.io/step_book_mcnulty/ch07/rotations-3d.html