

7.4 Matrices That Return to the Start

WORKSHEET

The question

A 2×2 matrix $\mathbf{M}$ is real if it can be written as $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c and d are real. In this case, the *trace* of matrix $\mathbf{M}$ is defined to be $\text{tr}(\mathbf{M}) = a + d$ and $\det(\mathbf{M})$ is the determinant of matrix $\mathbf{M}$. In this question, $\mathbf{M}$ is a real 2×2 matrix.

(i) Prove that

$$\text{tr}(\mathbf{M}^2) = \text{tr}(\mathbf{M})^2 - 2 \det(\mathbf{M}).$$

(ii) Prove that

$$\mathbf{M}^2 = \mathbf{I} \text{ but } \mathbf{M} \neq \pm \mathbf{I} \iff \text{tr}(\mathbf{M}) = 0 \text{ and } \det(\mathbf{M}) = -1,$$

and that

$$\mathbf{M}^2 = -\mathbf{I} \iff \text{tr}(\mathbf{M}) = 0 \text{ and } \det(\mathbf{M}) = 1.$$

(iii) Use part (ii) to prove that

$$\mathbf{M}^4 = \mathbf{I} \iff \mathbf{M}^2 = \pm \mathbf{I}.$$

Find a necessary and sufficient condition on $\det(\mathbf{M})$ and $\text{tr}(\mathbf{M})$ so that $\mathbf{M}^4 = -\mathbf{I}$.

(iv) Give an example of a matrix $\mathbf{M}$ for which $\mathbf{M}^8 = \mathbf{I}$, but which does not represent a rotation or reflection. [Note that the matrices $\pm \mathbf{I}$ are both rotations.]

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

Ask yourself Every 2×2 matrix satisfies $\mathbf{M}^2 - (\text{tr } \mathbf{M})\mathbf{M} + (\det \mathbf{M})\mathbf{I} = \mathbf{0}$ (the Cayley–Hamilton theorem; check it by multiplying out). How could that help with questions about $\mathbf{M}^2$?

STAGE 2 Square roots of $\pm \mathbf{I}$

Ask yourself Suppose $\mathbf{M}^2 = \mathbf{I}$. Substitute into $\mathbf{M}^2 = t\mathbf{M} - d\mathbf{I}$. If $t \neq 0$, what does that say about $\mathbf{M}$?

STAGE 3 Fourth roots

Pause and try Let $\mathbf{N} = \mathbf{M}^2$, so $\mathbf{M}^4 = \mathbf{I}$ means $\mathbf{N}^2 = \mathbf{I}$. By part (ii), either $\mathbf{N} = \pm \mathbf{I}$ or $\text{tr } \mathbf{N} = 0$ and $\det \mathbf{N} = -1$. Why is the second case impossible?

Try this next

- Rotations of order three.** (From STEP II, 2021.) The matrix $\mathbf{R}$ represents an anti-clockwise rotation through angle ϕ , and $\mathbf{R} + \mathbf{I}$ also represents a rotation. Find the possible values of ϕ and deduce that $\mathbf{R}^3 = \mathbf{I}$. Then show that if $\mathbf{S}$ is a real matrix with $\mathbf{S}^3 = \mathbf{I}$ but $\mathbf{S} \neq \mathbf{I}$, then $\det \mathbf{S} = 1$ and $\text{tr } \mathbf{S} = -1$.
- Order six.** Find a condition on the trace and determinant for $\mathbf{M}^6 = \mathbf{I}$ but $\mathbf{M}^2 \neq \mathbf{I}$ and $\mathbf{M}^3 \neq \mathbf{I}$.
- The crystallographic restriction.** Show that if $\mathbf{M}$ has integer entries, $\det \mathbf{M} = 1$ and $\mathbf{M}^n = \mathbf{I}$ for some n , then $\text{tr } \mathbf{M} \in \{-2, -1, 0, 1, 2\}$. Find an integer matrix of order 6.
- Cayley–Hamilton for powers.** Use $\mathbf{M}^2 = t\mathbf{M} - d\mathbf{I}$ to express $\mathbf{M}^3$ and $\mathbf{M}^4$ in the form $p\mathbf{M} + q\mathbf{I}$, and use this to rederive the condition for $\mathbf{M}^4 = -\mathbf{I}$.

Further reading

- **M. A. Armstrong, *Groups and Symmetry*** (Springer). Includes the crystallographic restriction and the classification of wallpaper patterns.
- **Marjorie Senechal, *Quasicrystals and Geometry*** (Cambridge University Press). The mathematics of quasicrystals and aperiodic order.
- **Gilbert Strang, *Introduction to Linear Algebra*** (Wellesley-Cambridge). Similar matrices

and the Cayley–Hamilton theorem.

The full guided solution, and what the question leads to at university, is in Section 7.4:
https://keithmcnulty.github.io/step_book_mcnulty/ch07/finite-order.html