

7.3 What Makes a Determinant

WORKSHEET

The question

The domain of the function f is the set of all 2×2 matrices and its range is the set of real numbers. Thus, if $\mathbf{M}$ is a 2×2 matrix, then $f(\mathbf{M}) \in \mathbb{R}$.

The function f has the property that $f(\mathbf{MN}) = f(\mathbf{M})f(\mathbf{N})$ for any 2×2 matrices $\mathbf{M}$ and $\mathbf{N}$.

- (i) You are given that there is a matrix $\mathbf{M}$ such that $f(\mathbf{M}) \neq 0$. Let $\mathbf{I}$ be the 2×2 identity matrix. By considering $f(\mathbf{MI})$, show that $f(\mathbf{I}) = 1$.
- (ii) Let $\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. You are given that $f(\mathbf{J}) \neq 1$. By considering $\mathbf{J}^2$, evaluate $f(\mathbf{J})$.

Using $\mathbf{J}$, show that, for any real numbers a, b, c and d ,

$$f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -f\begin{pmatrix} c & d \\ a & b \end{pmatrix} = f\begin{pmatrix} d & c \\ b & a \end{pmatrix}.$$

- (iii) Let $\mathbf{K} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$ where $k \in \mathbb{R}$. Use $\mathbf{K}$ to show that, if the second row of the matrix $\mathbf{A}$ is a multiple of the first row, then $f(\mathbf{A}) = 0$.
- (iv) Let $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. By considering the matrices $\mathbf{P}^2$, $\mathbf{P}^{-1}$, and $\mathbf{K}^{-1}\mathbf{PK}$ for suitable values of k , evaluate $f(\mathbf{P})$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

Ask yourself What does each special matrix in the question do? Work out $\mathbf{JM}$, $\mathbf{MJ}$, $\mathbf{KM}$ and $\mathbf{PM}$ for a general $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

STAGE 3 Dependent rows

Ask yourself First consider a matrix $\mathbf{B}$ with two *equal* rows. What is $\mathbf{JB}$? What does that say about $f(\mathbf{B})$?

STAGE 4 The shear

Pause and try Compute $\mathbf{P}^2$ and $\mathbf{K}^{-1}\mathbf{PK}$. Choose k to make them equal. What does that give for $f(\mathbf{P})$?

Try this next

- Other examples.** Show that $f(\mathbf{M}) = (\det \mathbf{M})^3$ satisfies all the conditions of the question, and that $f(\mathbf{M}) = |\det \mathbf{M}|$ satisfies the multiplicative property but not $f(\mathbf{J}) \neq 1$.
- Column operations.** Show that f is also zero on any matrix whose second column is a multiple of the first.
- Scalings.** Let $\mathbf{K}_k = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$. Show that $f\left(\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}\right) = f(\mathbf{K}_k)$, and that $f(\mathbf{K}_k)f(\mathbf{K}_l) = f(\mathbf{K}_{kl})$.
- Reduction to the determinant.** **HARDER** Show that every invertible 2×2 matrix can be written as a product of matrices of the forms $\mathbf{J}$, $\mathbf{K}_k$ and $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. Deduce that $f(\mathbf{M})$ depends only on $\det \mathbf{M}$.

Further reading

- **Sheldon Axler, *Linear Algebra Done Right*** (Springer; free online). Defines the determinant from its properties, in the spirit of this question.
- **Gilbert Strang, *Introduction to Linear Algebra*** (Wellesley-Cambridge). Chapter 5 devel-

ops determinants from three rules.

- **3Blue1Brown**, “**The determinant**” (video, free online). The area interpretation, beautifully animated.

The full guided solution, and what the question leads to at university, is in Section 7.3:
https://keithmcnulty.github.io/step_book_mcnulty/ch07/determinant-axioms.html