

7.2 Stretching Without Turning

WORKSHEET

The question

The transformation T from $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} X \\ Y \end{pmatrix}$ is given by

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{2}{5} \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Show that T leaves the vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ unchanged in direction but multiplied by a scalar, and that $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is similarly transformed.

The circle C whose equation is $x^2 + y^2 = 1$ transforms under T to a curve E . Show that E has equation

$$8X^2 + 12XY + 17Y^2 = 80,$$

and state the area of the region bounded by E . Show also that the greatest value of X on E is $2\sqrt{17/5}$.

Find the equation of the tangent to E at the point which corresponds to the point $\frac{1}{5}(3, 4)$ on C .

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself What does it mean for T to leave a vector “unchanged in direction but multiplied by a scalar”? Write it as an equation.

STAGE 2 The image of the circle



Ask yourself To find the equation of E , express x and y in terms of X and Y (invert the matrix), and substitute into $x^2 + y^2 = 1$.



Ask yourself What shape is E , and what is its area? Use the eigenvectors.

STAGE 3 The greatest value of X



Ask yourself On C , $X = \frac{2}{5}(9x - 2y)$. How large can $9x - 2y$ be when $x^2 + y^2 = 1$?

STAGE 4 A tangent



Pause and try The point $\frac{1}{5}(3, 4)$ on C has tangent line $3x + 4y = 5$. A linear map sends tangent lines to tangent lines. Use this to find the tangent to E .

Try this next

- Finding eigenvalues.** Show that the eigenvalues of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are the roots of $\lambda^2 - (a + d)\lambda + (ad - bc) = 0$, and check that the matrix in the question gives $\lambda = 2$ and 4 .
- Perpendicular eigenvectors.** Show that if $\mathbf{M}$ is symmetric and $\mathbf{M}\mathbf{u} = \lambda\mathbf{u}$, $\mathbf{M}\mathbf{v} = \mu\mathbf{v}$ with $\lambda \neq \mu$, then $\mathbf{u} \cdot \mathbf{v} = 0$. (*Hint:* compute $\mathbf{v} \cdot \mathbf{M}\mathbf{u}$ in two ways.)
- A rotation.** Show that the rotation matrix $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ has no real eigenvectors unless θ is a multiple of π . What are its complex eigenvalues?
- Powers.** Using the eigenvectors, find $T^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. (*Hint:* write $(1, 0)$ as a combination of $(1, 2)$ and $(2, -1)$.)

Further reading

- ▶ **Gilbert Strang**, *Introduction to Linear Algebra* (Wellesley-Cambridge). Chapter 6 is an excellent introduction to eigenvalues and their applications.
- ▶ **Kurt Bryan & Tanya Leise**, “The \$25,000,000,000 eigenvector: the linear algebra behind Google”, *SIAM Review* 48 (2006). Readable and freely available.
- ▶ **3Blue1Brown**, *Essence of Linear Algebra* (video series, free online). Beautiful animations of exactly the geometric ideas in this section.

The full guided solution, and what the question leads to at university, is in Section 7.2:
https://keithmcnulty.github.io/step_book_mcnulty/ch07/eigenvectors.html