

7.1 Rules That Fail for Matrices

WORKSHEET

The question

In this question, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{X}$ are non-zero 2×2 real matrices.

Are the following assertions true or false? You must provide a proof or a counterexample in each case.

- (i) If $\mathbf{AB} = \mathbf{0}$ then $\mathbf{BA} = \mathbf{0}$.
- (ii) $(\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2$.
- (iii) The equation $\mathbf{AX} = \mathbf{0}$ has a non-zero solution $\mathbf{X}$ if and only if $\det \mathbf{A} = 0$.
- (iv) For any $\mathbf{A}$ and $\mathbf{B}$ there are at most two matrices $\mathbf{X}$ such that $\mathbf{X}^2 + \mathbf{AX} + \mathbf{B} = \mathbf{0}$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

? **Ask yourself** For real numbers, why is it true that $ab = 0$ implies $a = 0$ or $b = 0$? Which property of numbers does the argument use, and does it hold for matrices?

STAGE 1 Can $\mathbf{AB} = \mathbf{0}$ but $\mathbf{BA} \neq \mathbf{0}$?

? **Ask yourself** Think geometrically. You want $\mathbf{B}$ to send everything into a line that $\mathbf{A}$ then squashes to zero. But in the other order, $\mathbf{A}$ should not squash everything that $\mathbf{B}$ then kills.

STAGE 3 Non-zero solutions of $\mathbf{AX} = \mathbf{0}$

? **Ask yourself** If $\det \mathbf{A} \neq 0$, what can you do to $\mathbf{AX} = \mathbf{0}$? If $\det \mathbf{A} = 0$, can you find a non-zero column vector $\mathbf{v}$ with $\mathbf{A}\mathbf{v} = \mathbf{0}$?

STAGE 4 How many square roots?

🔧 **Pause and try** Find non-zero $\mathbf{A}$ and $\mathbf{B}$ for which $\mathbf{X}^2 + \mathbf{AX} + \mathbf{B} = \mathbf{0}$ has infinitely many solutions. (*Hint*: make the left-hand side a perfect square, $(\mathbf{X} - \mathbf{I})^2$, and think about matrices whose square is zero.)

Try this next

- 1 When do they commute?** Find all 2×2 matrices that commute with $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
- 2 Square roots of zero.** Find all 2×2 real matrices $\mathbf{N}$ with $\mathbf{N}^2 = \mathbf{0}$. (*Hint*: use the Cayley–Hamilton theorem to show $\text{tr } \mathbf{N} = \det \mathbf{N} = 0$.)
- 3 Reflections.** Show that every reflection matrix $\begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}$ squares to $\mathbf{I}$, and find the line it reflects in.
- 4 Cayley–Hamilton.** Prove that every 2×2 matrix satisfies $\mathbf{M}^2 - (\text{tr } \mathbf{M})\mathbf{M} + (\det \mathbf{M})\mathbf{I} = \mathbf{0}$ by direct calculation.

Further reading

- ▶ **Gilbert Strang**, *Introduction to Linear Algebra* (Wellesley-Cambridge). The most popular first linear algebra text, with an emphasis on understanding.
- ▶ **Sheldon Axler**, *Linear Algebra Done Right* (Springer; free online). A more abstract approach, centred on linear maps rather than matrices.
- ▶ **Jim Baggott**, *The Quantum Story* (Oxford University Press). The history of quantum mechanics, including Heisenberg's matrix mechanics.

The full guided solution, and what the question leads to at university, is in Section 7.1:
https://keithmcnulty.github.io/step_book_mcnulty/ch07/matrix-rules.html