

6.8 A Group of Singular Matrices

WORKSHEET

The question

- (i) Let S be the set of matrices of the form

$$\begin{pmatrix} a & a \\ a & a \end{pmatrix},$$

where a is any real non-zero number. Show that S is closed under matrix multiplication and, further, that S is a group under matrix multiplication.

- (ii) Let G be a set of $n \times n$ matrices which is a group under matrix multiplication, with identity element $\mathbf{E}$. By considering equations of the form $\mathbf{BC} = \mathbf{D}$ for suitable elements $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ of G , show that if a given element $\mathbf{A}$ of G is a singular matrix (i.e. $\det \mathbf{A} = 0$), then all elements of G are singular. Give, with justification, an example of such a group of singular matrices in the case $n = 3$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

? **Ask yourself** In part (i), the identity matrix $\mathbf{I}$ is not of the form $\begin{pmatrix} a & a \\ a & a \end{pmatrix}$. So what could the identity element of S be?

STAGE 2 The group axioms

? **Ask yourself** Matrix multiplication is associative. For the identity, solve $\mathbf{M}_e \mathbf{M}_a = \mathbf{M}_a$. For inverses, solve $\mathbf{M}_a \mathbf{M}_b = \mathbf{M}_e$.

STAGE 3 Singularity spreads

! **Pause and try** Suppose $\mathbf{A} \in G$ is singular, and let $\mathbf{B}$ be any element of G . Write $\mathbf{B}$ as a product involving $\mathbf{A}$, and take determinants.

STAGE 4 A 3×3 example

? **Ask yourself** Can you generalise part (i) to 3×3 matrices? Or find something even simpler, using a matrix that “ignores” two of the three coordinates?

Try this next

- 1 An isomorphism.** Show that the map $\mathbf{M}_a \mapsto 2a$ is a one-to-one correspondence from S to the non-zero reals that turns matrix multiplication into multiplication of numbers.
- 2 Not a group.** Show that the set of matrices $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ with $a \neq 0$ is closed under multiplication but is not a group. (*Hint:* look for an identity element. Is there a left identity? A right identity?)
- 3 Idempotents.** Find all 2×2 real matrices $\mathbf{E}$ with $\mathbf{E}^2 = \mathbf{E}$. Describe them geometrically.
- 4 All entries equal.** Show that the $n \times n$ matrices with all entries equal to $a \neq 0$ form a group under multiplication, and find its identity element.

Further reading

- ▶ **M. A. Armstrong, *Groups and Symmetry*** (Springer). Matrix groups and isomorphisms, with many geometric examples.
- ▶ **Gilbert Strang, *Introduction to Linear Algebra*** (Wellesley-Cambridge). Projections and idempotent matrices, clearly explained.
- ▶ **Joseph Gallian, *Contemporary Abstract Algebra*** (CRC Press). A friendly first course with many examples of unusual groups.

The full guided solution, and what the question leads to at university, is in Section 6.8:
https://keithmcnulty.github.io/step_book_mcnulty/ch06/singular-group.html