6.7 The Quaternion Group

WORKSHEET

The question

The set S consists of ordered pairs of complex numbers (z_1, z_2) and a binary operation $\circ$ on S is defined by

$$(z_1, z_2) \circ (w_1, w_2) = \left(z_1 w_1 - z_2 w_2^*,\ z_1 w_2 + z_2 w_1^*\right).$$

Show that the operation $\circ$ is associative and determine whether it is commutative. Evaluate $(z,0) \circ (w,0)$, $(z,0) \circ (0,w)$, $(0,z) \circ (w,0)$ and $(0,z) \circ (0,w)$.

The set S_1 is the subset of S consisting of $A, B, \ldots, H$, where $A = (1,0)$, $B = (0,1)$, $C = (\mathrm{i},0)$, $D = (0,\mathrm{i})$, $E = (-1,0)$, $F = (0,-1)$, $G = (-\mathrm{i},0)$ and $H = (0,-\mathrm{i})$. Show that S_1 is closed under $\circ$ and that it has an identity element. Determine the inverse and order of each element of S_1. Show that S_1 is a group under $\circ$. [You are not required to compute the multiplication table in full.]

Show that $\{A, B, E, F\}$ is a subgroup of S_1 and determine whether it is isomorphic to the group generated by the 2×2 matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ under matrix multiplication.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

Ask yourself Think of the pair (z_1, z_2) as standing for a "number" $z_1 + z_2\mathrm{j}$, where j is a new symbol with $\mathrm{j}^2 = -1$ and $\mathrm{j}w = w^*\mathrm{j}$ for complex w. Multiply $(z_1 + z_2\mathrm{j})(w_1 + w_2\mathrm{j})$ using these rules. Do you get the formula for $\circ$?

STAGE 1 Associativity via matrices

Ask yourself Associate to (z_1, z_2) the complex matrix $M = \begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{pmatrix}$. What is the top row of the matrix product $M(z_1, z_2)\, M(w_1, w_2)$?

STAGE 3 The group of eight

Ask yourself Use the four special products to show that S_1 is closed. What is the identity?

Pause and try Find the order and inverse of each element.

Try this next

1. **Hamilton's rules.** Using the pairs, check that $C \circ B = D$ and $B \circ C = H$, that is, $\mathrm{ij} = \mathrm{k}$ and $\mathrm{ji} = -\mathrm{k}$.

2. **Only one element of order 2.** Show that $E = -1$ is the only element of order 2 in Q_8. Deduce that Q_8 is not isomorphic to the group of symmetries of a square (which has five elements of order 2).

3. **Four squares.** Show that $\det M(z_1, z_2) = |z_1|^2 + |z_2|^2$. Using $\det(M_1 M_2) = \det M_1 \det M_2$, deduce that a product of two sums of four squares is a sum of four squares.

4. **Subgroups.** Find all the subgroups of Q_8. Show that every subgroup contains E unless it is $\{A\}$, and that Q_8 is the union of its three cyclic subgroups of order 4 (compare Section 6.6).

Further reading

- **John Conway & Derek Smith, *On Quaternions and Octonions*** (A. K. Peters). The geome-

try and arithmetic of quaternions and octonions. Advanced but fascinating.

- **John Baez, "The octonions"**, *Bulletin of the American Mathematical Society* 39 (2002); also freely available online. A wonderful survey; the introduction is readable now.
- **Andrew Hanson, *Visualizing Quaternions*** (Morgan Kaufmann). Quaternions for rotations, with applications to graphics.

The full guided solution, and what the question leads to at university, is in Section 6.7:
https://keithmcnulty.github.io/step_book_mcnulty/ch06/quaternions.html