

6.6 Unions of Subgroups

WORKSHEET

The question

Give a careful argument to show that, if G_1 and G_2 are subgroups of a finite group G such that every element of G is either in G_1 or in G_2 , then either $G_1 = G$ or $G_2 = G$.

Give an example of a group H which has three subgroups H_1 , H_2 and H_3 such that every element of H is either in H_1 , H_2 or H_3 and $H_1 \neq H$, $H_2 \neq H$, $H_3 \neq H$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

 **Ask yourself** If K is a subgroup and $a \in K$, but $b \notin K$, can ab be in K ?

STAGE 1 Two subgroups cannot cover

 **Ask yourself** Since $G_1 \neq G$, there is an element not in G_1 . Where must it be? Similarly for G_2 . Now multiply the two elements together.

STAGE 2 Three subgroups can

 **Pause and try** Find a group that is the union of three subgroups, none of them the whole group. (*Hint*: try a small group in which every element is its own inverse.)

Try this next

- 1 Check the group.** Write out the multiplication table of the Klein four-group and check the group axioms. Show that it is not the same as the group $\{1, i, -1, -i\}$ under multiplication. (*Hint*: which elements are their own inverses?)
- 2 Cyclic groups.** A group is *cyclic* if every element is a power of a single element g . Show that a cyclic group is never the union of proper subgroups, however many. (*Hint*: which subgroup contains g ?)
- 3 The whole numbers.** Show directly that $(\mathbb{Z}, +)$ is not the union of the subgroups $2\mathbb{Z}$ (even numbers) and $3\mathbb{Z}$ (multiples of 3). Is it the union of any two proper subgroups?
- 4 Lines in the plane.** **HARDER** Show that the plane $\mathbb{R}^2$ is not the union of finitely many lines through the origin. (*Hint*: each line contains at most one point of the parabola $y = x^2$ other than the origin.)

Further reading

- **M. A. Armstrong, *Groups and Symmetry*** (Springer). A beautifully illustrated first course, driven by geometric examples.
- **Nathan Carter, *Visual Group Theory*** (Mathematical Association of America). Groups through pictures, ideal before university.

- **Peter Neumann, Gabrielle Stoy & Edward Thompson, *Groups and Geometry*** (Oxford University Press). A classic Oxford first-year text.

The full guided solution, and what the question leads to at university, is in Section 6.6:
https://keithmcnulty.github.io/step_book_mcnulty/ch06/subgroup-unions.html