6.5 Differentiation from Axioms

WORKSHEET

The question

$\Diamond$ is an operation which takes polynomials in x to polynomials in x; that is, given a polynomial $\mathrm{h}(x)$ there is another polynomial called $\Diamond\mathrm{h}(x)$. It is given that, if $\mathrm{f}(x)$ and $\mathrm{g}(x)$ are any two polynomials in x, the following are always true:

(i) $\Diamond\big(\mathrm{f}(x)\mathrm{g}(x)\big) = \mathrm{g}(x)\Diamond\mathrm{f}(x) + \mathrm{f}(x)\Diamond\mathrm{g}(x)$,

(ii) $\Diamond\big(\mathrm{f}(x) + \mathrm{g}(x)\big) = \Diamond\mathrm{f}(x) + \Diamond\mathrm{g}(x)$,

(iii) $\Diamond x = 1$,

(iv) if λ is a constant then $\Diamond\big(\lambda\mathrm{f}(x)\big) = \lambda\Diamond\mathrm{f}(x)$.

Show that, if $\mathrm{f}(x)$ is a constant (i.e., a polynomial of degree zero), then $\Diamond\mathrm{f}(x) = 0$.

Calculate $\Diamond x^2$ and $\Diamond x^3$. Prove that $\Diamond\mathrm{h}(x) = \dfrac{\mathrm{d}}{\mathrm{d}x}\big(\mathrm{h}(x)\big)$ for any polynomial $\mathrm{h}(x)$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

Ask yourself Rule (i) is the product rule. How could the product rule tell you anything about a constant? (Think of a constant as a product.)

STAGE 1 Constants

Ask yourself Apply rule (i) with $\mathrm{f} = \mathrm{g} = 1$. Then use rule (iv) for any other constant.

STAGE 2 Powers

Pause and try Prove by induction that $\Diamond x^n = nx^{n-1}$ for every positive integer n.

Try this next

1. **A different derivation.** Suppose $\Diamond$ satisfies rules (i), (ii) and (iv), but $\Diamond x = x$ instead of 1. Show that $\Diamond \mathrm{h}(x) = x\,\mathrm{h}'(x)$. What does $\Diamond$ do to x^n?

2. **The quotient rule.** Extend $\Diamond$ to rational functions f/g, assuming rule (i) still holds. By applying rule (i) to $\mathrm{g} \cdot \frac{\mathrm{f}}{\mathrm{g}} = \mathrm{f}$, derive the quotient rule.

3. **Repeated roots.** Show that if $(x-a)^2$ divides a polynomial f, then $x-a$ divides f'. Use this to show that $x^n - 1$ has no repeated roots.

4. **Dual numbers.** Use dual numbers to compute $f(2)$ and $f'(2)$ for $f(x) = (x^2+1)^3$, and check with the chain rule.

Further reading

- **Peter Cameron, *Introduction to Algebra*** (Oxford University Press). Rings, polynomials and fields at first-year level, with the formal derivative.
- **Andreas Griewank & Andrea Walther, *Evaluating Derivatives*** (SIAM). The standard reference on automatic differentiation. Advanced.
- **Michael Nielsen, *Neural Networks and Deep Learning*** (free online). Chapter 2 explains backpropagation clearly.

The full guided solution, and what the question leads to at university, is in Section 6.5:
https://keithmcnulty.github.io/step_book_mcnulty/ch06/derivations.html