

6.4 Solving the Quartic

WORKSHEET

The question

Let α , β , γ and δ be the roots of the quartic equation

$$x^4 + px^3 + qx^2 + rx + s = 0.$$

You are given that, for any such equation, $\alpha\beta + \gamma\delta$, $\alpha\gamma + \beta\delta$ and $\alpha\delta + \beta\gamma$ satisfy a cubic equation of the form

$$y^3 + Ay^2 + (pr - 4s)y + (4qs - p^2s - r^2) = 0.$$

Determine A .

Now consider the quartic equation given by $p = 0$, $q = 3$, $r = -6$ and $s = 10$.

- (i) Find the value of $\alpha\beta + \gamma\delta$, given that it is the largest root of the corresponding cubic equation.
- (ii) Hence, using the values of q and s , find the value of $(\alpha + \beta)(\gamma + \delta)$ and the value of $\alpha\beta$ given that $\alpha\beta > \gamma\delta$.
- (iii) Using these results, and the values of p and r , solve the quartic equation.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself How many ways are there to split four objects $\alpha, \beta, \gamma, \delta$ into two pairs?

STAGE 1 The missing coefficient



Ask yourself In a cubic $y^3 + Ay^2 + By + C = 0$, what is the sum of the roots in terms of A ? And what is $(\alpha\beta + \gamma\delta) + (\alpha\gamma + \beta\delta) + (\alpha\delta + \beta\gamma)$ in terms of the quartic's coefficients?

STAGE 2 The specific cubic



Ask yourself Try small divisors of 84.

STAGE 4 Solving the quartic



Pause and try Find $\alpha + \beta$ and $\gamma + \delta$. You know their sum (from p) and their product (from Stage 3); to decide which is which, use r .

Try this next

- 1 Check the cubic.** Verify that $y_1y_2 + y_2y_3 + y_3y_1 = pr - 4s$, where $y_1 = \alpha\beta + \gamma\delta$ etc. (Expand and use $\sum \alpha = -p$, $\sum \alpha\beta\gamma = -r$, $\alpha\beta\gamma\delta = s$.)
- 2 The cubic as a sum of cubes.** (From STEP III, 2011.) Show that if $q^2 \neq 4p^3$ then $x^3 - 3px + q = a(x - \alpha)^3 + b(x - \beta)^3$, where α, β are the roots of $pt^2 - qt + p^2 = 0$. Hence show that one root of $x^3 - 24x + 48 = 0$ is $x = \frac{2(2 - 2^{1/3})}{1 - 2^{1/3}}$, and check this numerically.
- 3 Another quartic.** Use the method of this section to solve $x^4 - 4x^3 + 7x^2 - 6x + 2 = 0$. (One root of the resolvent cubic is a small integer.)
- 4 Counting permutations.** Of the 24 permutations of $\alpha, \beta, \gamma, \delta$, find all those that send the pairing $\{\alpha\beta, \gamma\delta\}$ to itself. Show that there are 8 of them. (They form a subgroup of S_4 .)

Further reading

- ▶ **Mario Livio**, *The Equation That Couldn't Be Solved* (Simon & Schuster). The story of the quintic and the birth of group theory, for a general reader.
- ▶ **Ian Stewart**, *Why Beauty Is Truth: A History of Symmetry* (Basic Books). From Babylonian equations to Galois and beyond.
- ▶ **Ian Stewart**, *Galois Theory* (CRC Press). The undergraduate text, with a historical introduction covering exactly this material.

The full guided solution, and what the question leads to at university, is in Section 6.4:
https://keithmcnulty.github.io/step_book_mcnulty/ch06/quartic.html