6.3 A Polynomial for $\sqrt{2}+\sqrt{3}+\sqrt{5}$

WORKSHEET

The question

(i) Show that, if $\left(x-\sqrt{2}\right)^2=3$, then $x^4-10x^2+1=0$.

Deduce that, if $\mathrm{f}(x)=x^4-10x^2+1$, then $\mathrm{f}\left(\sqrt{2}+\sqrt{3}\right)=0$.

(ii) Find a polynomial g of degree 8 with integer coefficients such that $\mathrm{g}\left(\sqrt{2}+\sqrt{3}+\sqrt{5}\right)=0$. Write your answer in a form without brackets.

(iii) Let a, b and c be the three roots of $t^3-3t+1=0$.

Find a polynomial h of degree 6 with integer coefficients such that $\mathrm{h}\left(a+\sqrt{2}\right)=0$, $\mathrm{h}\left(b+\sqrt{2}\right)=0$ and $\mathrm{h}\left(c+\sqrt{2}\right)=0$. Write your answer in a form without brackets.

(iv) Find a polynomial k with integer coefficients such that $\mathrm{k}\left(\sqrt[3]{2}+\sqrt[3]{3}\right)=0$. Write your answer in a form without brackets.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

Ask yourself To find a polynomial with $\sqrt{2}$ as a root, you square $x = \sqrt{2}$ to get rid of the square root. How might you get rid of two different square roots, one at a time?

STAGE 2 Three square roots

Ask yourself Let $x = \sqrt{2}+\sqrt{3}+\sqrt{5}$, so $x-\sqrt{5} = \sqrt{2}+\sqrt{3}$ is a root of f. Write $f(x-\sqrt{5}) = 0$ in the form $P(x)+\sqrt{5}\,Q(x) = 0$. How do you then eliminate $\sqrt{5}$?

STAGE 3 A cubic and a square root

Ask yourself If $x = a+\sqrt{2}$, then $x-\sqrt{2}$ is a root of t^3-3t+1. Substitute, separate the terms with and without $\sqrt{2}$, and square.

STAGE 4 Two cube roots

Pause and try Let $x = \sqrt[3]{2}+\sqrt[3]{3}$. Cube both sides. Can you write the result in terms of x and a single cube root?

Try this next

1. **A primitive element.** Let $y = \sqrt{2}+\sqrt{3}$. Show that $y^3 = 11\sqrt{2}+9\sqrt{3}$, and deduce that $\sqrt{2} = \frac{1}{2}(y^3-9y)$. Hence every number $p+q\sqrt{2}+r\sqrt{3}+s\sqrt{6}$ can be written as a polynomial in y.

2. **Mixed roots.** Find a polynomial of degree 6 with integer coefficients that has $\sqrt{2}+\sqrt[3]{3}$ as a root.

3. **Irreducibility.** Show that x^4-10x^2+1 has no rational roots and cannot be written as a product of two quadratics with integer coefficients. (*Hint:* try $(x^2+ax+b)(x^2-ax+c)$.) Deduce that $\sqrt{2}+\sqrt{3}$ has degree 4.

4. **Trisection.** Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, and deduce that $\cos 20^\circ$ is a root of $8x^3-6x-1=0$. Assuming this cubic has no rational roots (check this), explain why a 60° angle cannot be trisected with ruler and compasses.

Further reading

- **Ian Stewart, *Galois Theory*** (CRC Press). Field extensions, the tower law and the classical construction problems, written for undergraduates.
- **Richard Courant & Herbert Robbins, *What Is Mathematics?*** (Oxford University Press). Chapter 3 proves the impossibility of the Greek problems with no prerequisites.
- **Robin Hartshorne, *Geometry: Euclid and Beyond*** (Springer). Ruler-and-compass constructions treated with modern algebra.

The full guided solution, and what the question leads to at university, is in Section 6.3:
https://keithmcnulty.github.io/step_book_mcnulty/ch06/field-extensions.html