

6.2 Gauss's 17-gon

WORKSHEET

The question

Let $y = \cos \phi + \cos 2\phi$, where $\phi = \frac{2\pi}{5}$. Verify by direct substitution that y satisfies the quadratic equation $2y^2 = 3y + 2$ and deduce that the value of y is $-\frac{1}{2}$.

Let $\theta = \frac{2\pi}{17}$. Show that

$$\sum_{k=0}^{16} \cos k\theta = 0.$$

If $z = \cos \theta + \cos 2\theta + \cos 4\theta + \cos 8\theta$, show that the value of z is $-(1 - \sqrt{17})/4$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself Why does the first part, about $\cos \frac{2\pi}{5}$, appear in a question about the 17-gon?

STAGE 1 The pentagon



Ask yourself Expand $y^2 = (\cos \phi + \cos 2\phi)^2$, turn each product into a sum, and use $5\phi = 2\pi$ to reduce angles like 3ϕ and 4ϕ .

STAGE 3 Splitting the cosines into two groups



Ask yourself Let $z = \cos \theta + \cos 2\theta + \cos 4\theta + \cos 8\theta$, and let z' be the sum of the other four cosines, $\cos 3\theta + \cos 5\theta + \cos 6\theta + \cos 7\theta$. What is $z + z'$?



Pause and try Compute zz' . Expand it as 16 products $\cos a\theta \cos b\theta$, turn each into a sum of two cosines, and count how often each of $\cos \theta, \cos 2\theta, \dots, \cos 8\theta$ appears. (Reduce angles using $\cos k\theta = \cos(17-k)\theta$ and $\cos(-k\theta) = \cos k\theta$.)

Try this next

- The heptagon.** (From STEP II, 1995.) Let $\alpha = e^{2\pi i/7}$. Show that $\eta = \alpha + \alpha^2 + \alpha^4$ and $\eta' = \alpha^3 + \alpha^5 + \alpha^6$ satisfy $\eta + \eta' = -1$ and $\eta\eta' = 2$. Deduce that $\cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{8\pi}{7} = -\frac{1}{2}$, and that $\sin \frac{2\pi}{7} + \sin \frac{4\pi}{7} + \sin \frac{8\pi}{7} = \frac{\sqrt{7}}{2}$, explaining how you decide the sign.
- The pentagon.** Show that $\cos \frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$, and use it to describe a ruler-and-compasses construction of a regular pentagon.
- The other period.** Check numerically that $z' = \cos 3\theta + \cos 5\theta + \cos 6\theta + \cos 7\theta = \frac{-1-\sqrt{17}}{4}$.
- Generators.** Show that the powers of 3 modulo 17 give every non-zero residue, but the powers of 2 do not. Which residues do the powers of 2 give?

Further reading

- Ian Stewart, *Galois Theory* (CRC Press). Chapter on ruler-and-compass constructions

and the regular 17-gon.

- ▶ **Richard Courant & Herbert Robbins, *What Is Mathematics?*** (Oxford University Press). Chapter 3 is a beautiful elementary account of constructions, including the 17-gon.
- ▶ **Daniel Kehlmann, *Measuring the World*** (Quercus). A comic novel about Gauss and Alexander von Humboldt, for light relief.

The full guided solution, and what the question leads to at university, is in Section 6.2:
https://keithmcnulty.github.io/step_book_mcnulty/ch06/gauss-17gon.html