

5.8 A Hidden Distance in the Disc

WORKSHEET

The question

Let z and w be complex numbers. Use a diagram to show that $|z - w| \leq |z| + |w|$.
For any complex numbers z and w , E is defined by

$$E = zw^* + z^*w + 2|zw|.$$

- (i) Show that $|z - w|^2 = (|z| + |w|)^2 - E$, and deduce that E is real and non-negative.
- (ii) Show that $|1 - zw^*|^2 = (1 + |zw|)^2 - E$.

Hence show that, if both $|z| > 1$ and $|w| > 1$, then

$$\frac{|z - w|}{|1 - zw^*|} \leq \frac{|z| + |w|}{1 + |zw|}.$$

Does this inequality also hold if both $|z| < 1$ and $|w| < 1$?

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself How does $|z - w| \leq |z| + |w|$ follow from a diagram? (Think of z and $-w$ as vectors.)

STAGE 1 Expanding the moduli



Ask yourself Expand $|z - w|^2 = (z - w)(z^* - w^*)$. Compare with $(|z| + |w|)^2$.

STAGE 3 The inequality



Ask yourself Write $A = (|z| + |w|)^2$ and $B = (1 + |zw|)^2$. The inequality, squared, says $\frac{A-E}{B-E} \leq \frac{A}{B}$. When is that true?



Pause and try Show that $1 + |z||w| - |z| - |w|$ factorises. When is it non-negative?

Try this next

- 1 A numerical check.** Take $z = \frac{1}{2}$ and $w = -\frac{1}{2}$. Compute both sides of the inequality. Then compute the hyperbolic distances $d(z, 0)$, $d(0, w)$ and $d(z, w)$ and check the triangle inequality is an equality here. Why?
- 2 The mixed case.** Find z, w with $|z| < 1 < |w|$ for which the inequality fails.
- 3 An isometry.** Let $f(z) = \frac{z - a}{1 - a^*z}$ with $|a| < 1$. Show that $|f(z)| = 1$ whenever $|z| = 1$, and that $f(a) = 0$. Show that $\rho(f(z), f(w)) = \rho(z, w)$.
- 4 Infinity is near.** Show that the hyperbolic distance from 0 to r (with $0 < r < 1$) is $\ln \frac{1+r}{1-r}$, and that it tends to infinity as $r \rightarrow 1$. How close to the boundary must a point be to be hyperbolic distance 10 from the centre?

Further reading

- **Caroline Series, “Hyperbolic geometry” lecture notes** (University of Warwick, freely available). A clear first course, starting from the Poincaré disc.
- **Jeremy Gray, *Ideas of Space*** (Oxford University Press). The history of non-Euclidean geometry from Euclid to Einstein.

- **Tristan Needham**, *Visual Complex Analysis* (Oxford University Press). Chapter 6 covers non-Euclidean geometry and Möbius transformations, with beautiful pictures.

The full guided solution, and what the question leads to at university, is in Section 5.8:
https://keithmcnulty.github.io/step_book_mcnulty/ch05/poincare-disc.html