5.7 The Complex Exponential as a Map

WORKSHEET

The question

The complex numbers $w = u + \mathrm{i}v$ and $z = x + \mathrm{i}y$ are related by the equation

$$z = (\cos v + \mathrm{i}\sin v)\mathrm{e}^u.$$

Find all w which correspond to $z = \mathrm{i}\,\mathrm{e}$.

Find the loci in the x–y plane corresponding to the lines $u =$ constant in the u–v plane. Find also the loci corresponding to the lines $v =$ constant. Illustrate your answers with clearly labelled sketches.

Identify two subsets W_1 and W_2 of the u–v plane each of which is in one-to-one correspondence with the first quadrant $\{(x, y) : x > 0, y > 0\}$ of the x–y plane. Identify also two subsets W_3 and W_4 each of which is in one-to-one correspondence with the set $\{z : 0 < |z| < 1\}$.

[**NB** 'one-to-one' means here that to each value of w there is only one corresponding value of z, and vice-versa.]

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

Ask yourself What are the modulus and argument of $z = e^u(\cos v + \mathrm{i}\sin v)$ in terms of u and v?

STAGE 1 Solving $e^w = \mathrm{i}e$

Ask yourself What are the modulus and argument of $\mathrm{i}e$?

STAGE 3 One-to-one pieces

Ask yourself Which points w map into the first quadrant $x > 0$, $y > 0$? Why is that set too big to be in one-to-one correspondence with the quadrant?

Pause and try Now find two sets in the w-plane in one-to-one correspondence with the punctured disc $0 < |z| < 1$.

Try this next

1. **Solving equations.** Find all complex w with $e^w = -1$, and all w with $e^w = 1 + \mathrm{i}$.
2. **The image of a square.** Find and sketch the image under $z = e^w$ of the square with corners 0, 1, $1 + \mathrm{i}$ and i.
3. **Right angles.** Show directly that the circle $|z| = e^c$ and the ray $\arg z = d$ meet at right angles, and explain why this is consistent with the lines $u = c$ and $v = d$ being perpendicular.
4. **Squaring.** Find the images of the lines $x = c$ and $y = d$ under the map $z \mapsto z^2$. Show that they are parabolas, and that they meet at right angles.

Further reading

- **Tristan Needham, *Visual Complex Analysis*** (Oxford University Press). Chapters 2 and 4 treat the exponential, the logarithm and conformal maps with superb pictures.
- **Hilary Priestley, *Introduction to Complex Analysis*** (Oxford University Press). A clear first undergraduate text.
- **Ian Stewart & David Tall, *Complex Analysis*** (Cambridge University Press). Another

excellent undergraduate introduction, with an emphasis on geometry.

The full guided solution, and what the question leads to at university, is in Section 5.7:
https://keithmcnulty.github.io/step_book_mcnulty/ch05/complex-exponential.html