

5.6 The Fermat Point

WORKSHEET

The question

Given that $\alpha = e^{i\pi/3}$, prove that $1 + \alpha^2 = \alpha$.

A triangle in the Argand plane has vertices A , B , and C represented by the complex numbers p , $q\alpha^2$ and $-\alpha$ respectively, where p , q and r are positive real numbers. Sketch the triangle ABC .

Three equilateral triangles ABL , BCM and CAN (each lettered clockwise) are erected on sides AB , BC and CA respectively. Show that the complex number representing N is $(1 - \alpha)p - \alpha^2 r$ and find similar expressions for the complex numbers representing L and M .

Show that lines LC , MA and NB all meet at the origin, and that these three line segments have the common length $p + q + r$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself Where are A , B and C ? What are their arguments? What angles do OA , OB and OC make with each other?

STAGE 2 Equilateral triangles as rotations



Ask yourself Triangle CAN is equilateral, lettered clockwise. If you stand at C , how do you get from A to N ? Express $N - C$ in terms of $A - C$.



Pause and try Find L and M in the same way, and simplify using the identities from Stage 1.

STAGE 3 Three lines through the origin



Ask yourself Compare $L = \alpha(p + q)$ with $C = -r\alpha$. What do you notice? Do the same for M and A , and for N and B .

Try this next

- 1 Check the lettering.** Show that if the equilateral triangles were lettered anticlockwise (built inwards), the three lines would generally not pass through O .
- 2 Napoleon's theorem.** Show that the centre of the equilateral triangle ABL is $\frac{1}{3}(A + B + L)$. Using the expressions above, show that the three centres form an equilateral triangle.
- 3 The minimum.** With $p = 2$, $q = 1.4$, $r = 1.1$, compute $PA + PB + PC$ for P at each vertex and at the centroid $\frac{1}{3}(A + B + C)$, and check that all exceed 4.5.
- 4 An equilateral test.** Show that z_1, z_2, z_3 form an equilateral triangle (in either orientation) if and only if $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$.

Further reading

- **Richard Courant & Herbert Robbins, *What Is Mathematics?*** (Oxford University Press). Chapter 7 discusses the Fermat point, Steiner trees and soap films.
- **Liang-shin Hahn, *Complex Numbers and Geometry*** (Mathematical Association of America). Many geometric theorems proved with complex numbers, at exactly this

level.

- **Tristan Needham**, *Visual Complex Analysis* (Oxford University Press). Chapter 1 includes Napoleon's theorem and other geometric results.

The full guided solution, and what the question leads to at university, is in Section 5.6:
https://keithmcnulty.github.io/step_book_mcnulty/ch05/fermat-point.html