5.5 Rational Angles That Aren't

WORKSHEET

The question

(i) Use De Moivre's theorem to prove that for any positive integer $k > 1$,

$$\sin(k\theta) = \sin\theta\cos^{k-1}\theta\left(k - \binom{k}{3}(\sec^2\theta - 1) + \binom{k}{5}(\sec^2\theta - 1)^2 - \cdots\right)$$

and find a similar expression for $\cos(k\theta)$.

(ii) Let $\theta = \cos^{-1}\left(\frac{1}{a}\right)$, where θ is measured in degrees, and a is an odd integer greater than 1.

Suppose that there is a positive integer k such that $\sin(k\theta) = 0$ and $\sin(m\theta) \neq 0$ for all integers m with $0 < m < k$.

Show that it would be necessary to have k even and $\cos(\frac{1}{2}k\theta) = 0$.

Deduce that θ is irrational.

(iii) Show that if $\phi = \cot^{-1}\left(\frac{1}{b}\right)$, where ϕ is measured in degrees, and b is an even integer greater than 1, then ϕ is irrational.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

Ask yourself Suppose θ were rational, say $\theta = \frac{p}{q}$ degrees. Find a positive integer k with $\sin(k\theta) = 0$.

STAGE 1 Multiple-angle formulas

Ask yourself Write $\cos k\theta + \mathrm{i}\sin k\theta = (\cos\theta + \mathrm{i}\sin\theta)^k$, and take out a factor of $\cos^k\theta$. What is left inside the bracket?

STAGE 2 Parity

Ask yourself If $\sin(k\theta) = 0$, the bracket $k - \binom{k}{3}(a^2-1) + \cdots$ must be zero. Every term after the first is a multiple of $a^2 - 1$. What does that tell you about k?

STAGE 3 The contradiction

Pause and try Apply the cosine formula with $m = \frac{1}{2}k$. Why can $\cos(m\theta)$ never be zero?

STAGE 4 The cotangent version

Ask yourself If $\cot\phi = \frac{1}{b}$, what is $\sec^2\phi - 1$? Is it even?

Try this next

1. **The tetrahedron.** Show that the angle between two faces of a regular tetrahedron is $\cos^{-1}\frac{1}{3}$, and that the angle between two bonds in a methane molecule (from the centre of a regular tetrahedron to two vertices) is $\cos^{-1}\left(-\frac{1}{3}\right)$.

2. **The 3–4–5 triangle.** Show that the smallest angle of a 3–4–5 triangle is an irrational number of degrees. (*Hint:* it is $\cot^{-1}\frac{4}{3}$; adapt the argument, or consider powers of the Gaussian integer $4 + 3\mathrm{i}$ and show none is real.)

3. **Why odd a?** What goes wrong in the argument when $a = 2$? Is $\cos^{-1}\frac{1}{2}$ rational?

4. **Niven for small cases.** Find all whole numbers of degrees θ between 0° and 90° for

which $\cos\theta$ is rational, and check that your list agrees with Niven's theorem.

Further reading

- **Ivan Niven,** ***Irrational Numbers*** (Mathematical Association of America). Includes Niven's theorem on rational values of trigonometric functions.
- **Martin Aigner & Günter Ziegler,** ***Proofs from THE BOOK***, chapter on Hilbert's third problem. A beautiful account of Dehn's solution.
- **Benjamin Yandell,** ***The Honors Class: Hilbert's Problems and Their Solvers*** (A. K. Peters). The stories behind all twenty-three problems.

The full guided solution, and what the question leads to at university, is in Section 5.5:
https://keithmcnulty.github.io/step_book_mcnulty/ch05/rational-angles.html