5.4 Distances in a Regular Polygon

WORKSHEET

The question

Let $\omega = e^{2\pi i/n}$, where n is a positive integer. Show that, for any complex number z,

$$(z-1)(z-\omega)\cdots(z-\omega^{n-1}) = z^n - 1.$$

The points $X_0, X_1, \ldots, X_{n-1}$ lie on a circle with centre O and radius 1, and are the vertices of a regular polygon.

(i) The point P is equidistant from X_0 and X_1. Show that, if n is even,

$$|PX_0| \times |PX_1| \times \cdots \times |PX_{n-1}| = |OP|^n + 1,$$

where $|PX_k|$ denotes the distance from P to X_k.

Give the corresponding result when n is odd. (There are two cases to consider.)

(ii) Show that

$$|X_0X_1| \times |X_0X_2| \times \cdots \times |X_0X_{n-1}| = n.$$

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

Ask yourself Put the polygon's vertices at the nth roots of unity, $X_k = \omega^k$. If P is the point z, what is $|PX_k|$ in terms of z? So what is the product of all the distances?

STAGE 2 A point on the perpendicular bisector

Ask yourself So P is $z = te^{i\pi/n}$ for some real t (positive or negative), with $|OP| = |t|$. What is z^n?

Pause and try What happens when n is odd? Consider $t > 0$ and $t < 0$ separately, and say what each case means geometrically.

STAGE 3 Chords from a vertex

Ask yourself Now $P = X_0$, and one of the distances, $|X_0X_0|$, is zero, so $|z^n - 1| = 0$ is no use. What polynomial has the roots $\omega, \omega^2, \dots, \omega^{n-1}$ but not 1?

Try this next

1. **Check small cases.** Verify part (ii) directly for a square ($n = 4$) and an equilateral triangle ($n = 3$).

2. **Through a vertex.** Show that if P lies on the ray from O through X_0, then $|PX_0| \times \cdots \times |PX_{n-1}| = \big||OP|^n - 1\big|$.

3. **The average distance.** Using part (ii) and the AM–GM inequality of Section 3.2, show that the average length of the $n-1$ chords from X_0 is at least $n^{1/(n-1)}$. Find the actual average, $\frac{1}{n-1}\sum_k 2\sin\frac{k\pi}{n}$, for $n = 6$, and compare.

4. **All the chords.** HARDERShow that the product of all $\binom{n}{2}$ chord lengths of a regular n-gon inscribed in a unit circle is $n^{n/2}$.

Further reading

- **Tristan Needham, *Visual Complex Analysis*** (Oxford University Press). Geometry with complex numbers, beautifully illustrated.

- **Ross Honsberger, *Mathematical Gems*** (Mathematical Association of America). Several elegant problems about regular polygons.
- **Ronald Gowing, *Roger Cotes: Natural Philosopher*** (Cambridge University Press). A biography of Cotes and his mathematics.

The full guided solution, and what the question leads to at university, is in Section 5.4:
https://keithmcnulty.github.io/step_book_mcnulty/ch05/polygon-distances.html