

5.3 Factorising $z^{2n} + 1$

WORKSHEET

The question

Show that $(z - e^{i\theta})(z - e^{-i\theta}) = z^2 - 2z \cos \theta + 1$.

Write down the $(2n)$ th roots of -1 in the form $e^{i\theta}$, where $-\pi < \theta \leq \pi$, and deduce that

$$z^{2n} + 1 = \prod_{k=1}^n \left(z^2 - 2z \cos \left(\frac{(2k-1)\pi}{2n} \right) + 1 \right).$$

Here, n is a positive integer, and the $\prod$ notation denotes the product.

(i) By substituting $z = i$ show that, when n is even,

$$\cos \left(\frac{\pi}{2n} \right) \cos \left(\frac{3\pi}{2n} \right) \cos \left(\frac{5\pi}{2n} \right) \cdots \cos \left(\frac{(2n-1)\pi}{2n} \right) = (-1)^{\frac{1}{2}n} 2^{1-n}.$$

(ii) Show that, when n is odd,

$$\cos^2 \left(\frac{\pi}{2n} \right) \cos^2 \left(\frac{3\pi}{2n} \right) \cos^2 \left(\frac{5\pi}{2n} \right) \cdots \cos^2 \left(\frac{(n-2)\pi}{2n} \right) = n 2^{1-n}.$$

You may use without proof the fact that $1 + z^{2n} = (1 + z^2)(1 - z^2 + z^4 - \cdots + z^{2n-2})$ when n is odd.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

? **Ask yourself** Where are the solutions of $z^{2n} = -1$ in the complex plane? Draw them for $n = 2$ and $n = 4$.

STAGE 3 Substituting $z = i$

? **Ask yourself** What is $i^2 - 2i \cos \theta + 1$? And what is $i^{2n} + 1$ when n is even?

STAGE 4 When n is odd

? **Ask yourself** For odd n , one of the angles θ_k equals $\frac{\pi}{2}$. Which one, and why does that make the substitution $z = i$ useless as it stands?

🔧 **Pause and try** Divide both sides of the factorisation by $1 + z^2$, substitute $z = i$, and simplify. You will need to pair up the remaining cosines.

Try this next

1 A heptagonal product. Use part (ii) with $n = 7$ to show that $\cos \frac{\pi}{14} \cos \frac{3\pi}{14} \cos \frac{5\pi}{14} = \frac{\sqrt{7}}{8}$. Then show that $\cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{3\pi}{7} = \frac{1}{8}$. (*Hint:* relate the angles using $\cos \theta = \sin(\frac{\pi}{2} - \theta)$, or use the identity $\sin 2x = 2 \sin x \cos x$ repeatedly.)

2 Another substitution. Substitute $z = 1$ in the factorisation to show that

$$\prod_{k=1}^n \sin \frac{(2k-1)\pi}{4n} = 2^{\frac{1}{2}-n}.$$

3 Factorise $z^6 + 1$. Write $z^6 + 1$ as a product of three real quadratics, and check your answer by multiplying out.

4 Sines. By factorising $z^{2n} - 1$ similarly and dividing by $z^2 - 1$, show that

$$\prod_{k=1}^{n-1} \sin \frac{k\pi}{2n} = \frac{\sqrt{n}}{2^{n-1}}.$$

Further reading

- **Tristan Needham, *Visual Complex Analysis*** (Oxford University Press). Chapter 2 on roots of unity and polynomials, with beautiful pictures.

- ▶ **Benjamin Fine & Gerhard Rosenberger**, *The Fundamental Theorem of Algebra* (Springer). Several proofs of the theorem, from analysis, algebra and topology, written for undergraduates.
- ▶ **Paul Nahin**, *An Imaginary Tale* (Princeton University Press). A history of $\sqrt{-1}$, including roots of unity and their uses.

The full guided solution, and what the question leads to at university, is in Section 5.3:
https://keithmcnulty.github.io/step_book_mcnulty/ch05/z2n-plus-one.html