

5.2 The Roots-of-Unity Filter

WORKSHEET

The question

Let $\omega = e^{2\pi i/3}$. Show that $1 + \omega + \omega^2 = 0$ and calculate the modulus and argument of $1 + \omega^2$.

Let n be a positive integer. By evaluating $(1 + \omega^r)^n$ in two ways, taking $r = 1, 2$ and 3, or otherwise, prove that

$$\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \cdots + \binom{n}{k} = \frac{1}{3} \left(2^n + 2 \cos \left(\frac{n\pi}{3} \right) \right),$$

where k is the largest multiple of 3 less than or equal to n . Without using a calculator, evaluate

$$\binom{25}{0} + \binom{25}{3} + \cdots + \binom{25}{24}$$

and

$$\binom{24}{2} + \binom{24}{5} + \cdots + \binom{24}{23}.$$

[$2^{25} = 33554432$.]

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

? **Ask yourself** ω is a root of $z^3 - 1 = 0$. Factorise $z^3 - 1$. Why does this give $1 + \omega + \omega^2 = 0$?

STAGE 2 The filter

? **Ask yourself** What is $1 + \omega^k + \omega^{2k}$ when k is a multiple of 3? When it is not?

✓ **Pause and try** Add up $(1 + \omega^r)^n$ for $r = 1, 2, 3$, expanding each by the binomial theorem. What do you get?

STAGE 3 Two calculations

? **Ask yourself** For the second sum we need positions $k \equiv 2 \pmod{3}$. How can you modify the filter to detect $k \equiv 2$ instead of $k \equiv 0$?

Try this next

- 1 Even positions.** Use the filter with square roots of unity (1 and -1) to show that $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = 2^{n-1}$.
- 2 The other residue.** Find a formula for $\binom{n}{1} + \binom{n}{4} + \binom{n}{7} + \cdots$, and check that the three formulas add up to 2^n .
- 3 Every fourth.** Using i, show that $\binom{n}{0} + \binom{n}{4} + \binom{n}{8} + \cdots = \frac{1}{4} \left(2^n + 2^{n/2+1} \cos \frac{n\pi}{4} \right)$.
- 4 Divisibility.** How many subsets of $\{1, 2, \dots, 12\}$ have a number of elements divisible by 3? How many have a sum divisible by 3? (*Hint for the second:* consider $\prod_{j=1}^{12} (1 + x^j)$ at the cube roots of unity.)

Further reading

- **Herbert Wilf, *generatingfunctionology*** (A. K. Peters; free online). A delightful introduction to generating functions; the roots-of-unity filter appears early on.
- **Arthur Benjamin & Jennifer Quinn, *Proofs That Really Count*** (Mathematical Association of America). Binomial identities proved by counting.
- **T. W. Körner, *Fourier Analysis*** (Cambridge University Press). Includes the discrete

Fourier transform and the FFT.

The full guided solution, and what the question leads to at university, is in Section 5.2:
https://keithmcnulty.github.io/step_book_mcnulty/ch05/roots-filter.html