

5.1 Machin's Formula

WORKSHEET

The question

- (i) If $z = x + iy$, with x, y real, show that

$$|x| \cos \alpha + |y| \sin \alpha \leq |z| \leq |x| + |y|$$

for all real α .

- (ii) By considering $(5 - i)^4(1 + i)$, show that

$$\frac{\pi}{4} = 4 \tan^{-1} \left(\frac{1}{5} \right) - \tan^{-1} \left(\frac{1}{239} \right).$$

Prove similarly that

$$\frac{\pi}{4} = 3 \tan^{-1} \left(\frac{1}{4} \right) + \tan^{-1} \left(\frac{1}{20} \right) + \tan^{-1} \left(\frac{1}{1985} \right).$$

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself What is the argument of $5 - i$? Of $1 + i$? What happens to arguments when complex numbers are multiplied?

STAGE 1 Bounds on the modulus



Ask yourself For the upper bound, square both sides. For the lower bound, think of $|x| \cos \alpha + |y| \sin \alpha$ as a dot product of two vectors.

STAGE 2 The key multiplication



Ask yourself Compute $(5 - i)^2$, then square it, then multiply by $1 + i$.



Pause and try Now compare arguments. What is the argument of $956 - 4i$? Why are you sure there is no extra multiple of 2π ?

STAGE 3 The second formula



Ask yourself By analogy, which product should you compute this time?

Try this next

- 1 **A cousin.** (From STEP II, 2000.) By considering $(5 - i)^2(1 + i)$, prove that $\tan^{-1} \frac{7}{17} + 2 \tan^{-1} \frac{1}{5} = \frac{\pi}{4}$.
- 2 **Euler's formula.** Show that $\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \frac{\pi}{4}$ by considering $(2 + i)(3 + i)$.
- 3 **Hutton's formula.** Show that $\frac{\pi}{4} = 2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7}$.
- 4 **How many terms?** Estimate how many terms of each series in Machin's formula you would need to compute π to 20 decimal places. Then do it for the first few digits with a calculator.

Further reading

- **Jonathan Borwein & Peter Borwein, *Pi and the AGM*** (Wiley). Includes the history of Machin-type formulas.

- **Jörg Arndt & Christoph Haenel, *Pi Unleashed*** (Springer). A lively account of methods for computing π , old and new.
- **Tristan Needham, *Visual Complex Analysis*** (Oxford University Press). The best book for seeing complex numbers geometrically. Chapter 1 is about multiplication as rotation and scaling.

The full guided solution, and what the question leads to at university, is in Section 5.1:
https://keithmcnulty.github.io/step_book_mcnulty/ch05/machin.html