4.9 Holditch's Theorem

WORKSHEET

The question

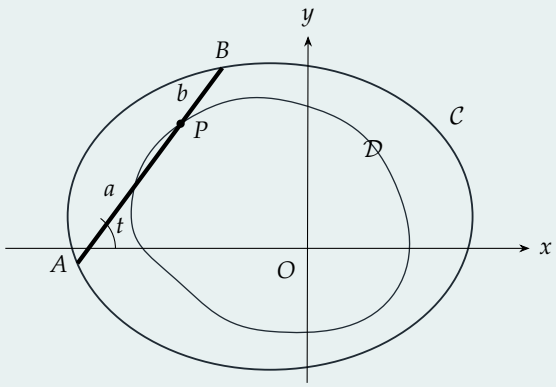
A movable point P has cartesian coordinates (x, y), where x and y are functions of t. The polar coordinates of P with respect to the origin O are r and θ. Starting with the expression

$$\tfrac{1}{2}\int r^2\,\mathrm{d}\theta$$

for the area swept out by OP, obtain the equivalent expression

$$\tfrac{1}{2}\int\left(x\frac{\mathrm{d}y}{\mathrm{d}t}-y\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t. \qquad (*)$$

The ends of a thin straight rod AB lie on a closed convex curve $\mathcal{C}$. The point P on the rod is a fixed distance a from A and a fixed distance b from B. The angle between AB and the positive x direction is t. As A and B move anticlockwise round $\mathcal{C}$, the angle t increases from 0 to 2π and P traces a closed convex curve $\mathcal{D}$ inside $\mathcal{C}$, with the origin O lying inside $\mathcal{D}$, as shown in the diagram.



Let (x, y) be the coordinates of P. Write down the coordinates of A and B in terms of a, b, x, y and t.

The areas swept out by OA, OB and OP are denoted by $[A]$, $[B]$ and $[P]$, respectively. Show, using $(*)$, that

$$[A] = [P] + \pi a^2 - af$$

where

$$f = \tfrac{1}{2}\int_0^{2\pi} \left(\left(x + \frac{\mathrm{d}y}{\mathrm{d}t} \right) \cos t + \left(y - \frac{\mathrm{d}x}{\mathrm{d}t} \right) \sin t \right) \mathrm{d}t.$$

Obtain a corresponding expression for $[B]$ involving b. Hence show that the area between the curves $\mathcal{C}$ and $\mathcal{D}$ is πab.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

Ask yourself Check the theorem in the simplest case. If $\mathcal{C}$ is a circle of radius R, what curve does P trace, and what is the area of the ring?

STAGE 1 Area from coordinates

Ask yourself With $x = r\cos\theta$ and $y = r\sin\theta$, all functions of t, compute $x\dot{y} - y\dot{x}$, where dots denote $\frac{\mathrm{d}}{\mathrm{d}t}$.

STAGE 3 The area swept by OA

Ask yourself Apply $(*)$ to A. Expand $x_A\dot{y}_A - y_A\dot{x}_A$ and sort the terms by powers of a.

Pause and try Now find $[B]$. (*Hint:* B is obtained from the formula for A by replacing a with what?)

STAGE 4 Eliminating f

Ask yourself Both A and B go once round the same curve $\mathcal{C}$. What does that tell you about $[A]$ and $[B]$?

Try this next

1. **The area of an ellipse.** Use $(*)$ with $x = p\cos t$, $y = q\sin t$ to show that an ellipse with semi-axes p and q has area πpq.

2. **The circle case.** Prove the intersecting chords theorem: if two chords of a circle meet at P, the products of the lengths of their two segments are equal. Deduce Holditch's theorem for a circle.

3. **The midpoint.** Show that $f = \pi(a - b)$ means that when P is the midpoint of the rod, $f = 0$. What is the ring's area then, in terms of the rod's length L?

4. **A triangle.** HARDERUse $(*)$ to find the area of the triangle with vertices $(0,0)$, (x_1, y_1), (x_2, y_2), traversing its boundary, and show it equals $\frac{1}{2}|x_1y_2 - x_2y_1|$. This is the start of the "shoelace formula" for the area of any polygon.

Further reading

- **Mark Levi, *The Mathematical Mechanic*** (Princeton University Press). Uses physical reasoning, with rods, wheels and springs, to prove mathematical results, in the spirit of this section.
- **H. M. Schey, *Div, Grad, Curl, and All That*** (Norton). An informal introduction to vector calculus, including Green's theorem.
- **Hamnet Holditch, "Geometrical theorem"**, *Quarterly Journal of Pure and Applied Mathematics* 2 (1858). The original note, a single short paragraph with none of the fine print.

The full guided solution, and what the question leads to at university, is in Section 4.9:
https://keithmcnulty.github.io/step_book_mcnulty/ch04/holditch.html