

4.8 Slicing a Spherical Loaf

WORKSHEET

The question

A spherical loaf of bread is cut into parallel slices of equal thickness. Show that, after any number of the slices have been eaten, the area of crust remaining is proportional to the number of slices remaining.

A European ruling decrees that a parallel-sliced spherical loaf can only be referred to as 'crusty' if the ratio of volume V (in cubic metres) of bread remaining to area A (in square metres) of crust remaining after any number of slices have been eaten satisfies $V/A < 1$. Show that the radius of a crusty parallel-sliced spherical loaf must be less than $2\frac{2}{3}$ metres.

[The area A and volume V formed by rotating a curve in the x - y plane round the x -axis from $x = -a$ to $x = -a + t$ are given by

$$A = 2\pi \int_{-a}^{-a+t} y \left(1 + \left(\frac{dy}{dx} \right)^2 \right)^{\frac{1}{2}} dx, \quad V = \pi \int_{-a}^{-a+t} y^2 dx.]$$

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself The sphere of radius a is obtained by rotating which curve about the x -axis? What is $\frac{dy}{dx}$ for that curve?

STAGE 1 The crust formula simplifies



Ask yourself Compute $y\sqrt{1+(y')^2}$ for the semicircle. Something remarkable happens.

STAGE 3 The crusty condition



Ask yourself Compute V using the given formula. A substitution $s = x + a$ might make the algebra cleaner.



Pause and try The ruling requires $V/A < 1$ after any number of slices have been eaten, that is, for every t with $0 < t \leq 2a$. What is the largest value of $\frac{t}{2} - \frac{t^2}{6a}$ on this range?

Try this next

- 1 The whole loaf.** Show that for the whole uncut loaf, $V/A = \frac{a}{3}$. Why is the condition in the question harder to satisfy than $\frac{a}{3} < 1$?
- 2 The tropics.** What fraction of the Earth's surface lies between latitudes 23.4° south and 23.4° north? (Treat the Earth as a sphere. Since area is proportional to height, you only need $\sin 23.4^\circ$.)
- 3 Other shapes.** Show that for a loaf in the shape of a paraboloid (rotate $y = \sqrt{x}$ about the x -axis), the crust per slice is *not* constant. For which curves $y(x)$ is $y\sqrt{1+(y')^2}$ constant? (Solve the differential equation.)
- 4 Volumes.** Use the volume formula to show that a sphere has two-thirds the volume of its wrapping cylinder, as Archimedes found.

Further reading

- **Sherman Stein, *Archimedes: What Did He Do Besides Cry Eureka?*** (Mathematical Association of America). Archimedes' mathematics explained for a modern reader.

- ▶ **Timothy Gowers, *Mathematics: A Very Short Introduction*** (Oxford University Press). The chapter on geometry introduces the geometry of curved surfaces.
- ▶ **John Snyder, *Flattening the Earth*** (University of Chicago Press). Two thousand years of map projections.

The full guided solution, and what the question leads to at university, is in Section 4.8:
https://keithmcnulty.github.io/step_book_mcnulty/ch04/spherical-loaf.html