4.7 The Calculus of Variations

WORKSHEET

The question

(i) Let

$$I = \int_0^1 \left((y')^2 - y^2\right) \mathrm{d}x \qquad \text{and} \qquad I_1 = \int_0^1 (y' + y\tan x)^2 \, \mathrm{d}x,$$

where y is a given function of x satisfying $y = 0$ at $x = 1$. Show that $I - I_1 = 0$ and deduce that $I \geq 0$. Show further that $I = 0$ only if $y = 0$ for all x $(0 \leq x \leq 1)$.

(ii) Let

$$J = \int_0^1 \left((y')^2 - a^2y^2\right) \mathrm{d}x,$$

where a is a given positive constant and y is a given function of x, not identically zero, satisfying $y = 0$ at $x = 1$. By considering an integral of the form

$$\int_0^1 (y' + ay\tan bx)^2 \, \mathrm{d}x,$$

where b is suitably chosen, show that $J \geq 0$. You should state the range of values of a, in the form $a < k$, for which your proof is valid.

In the case $a = k$, find a function y (not everywhere zero) such that $J = 0$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

Ask yourself Why is it not obvious that $I \geq 0$? Try $y = 1 - x$, which satisfies $y(1) = 0$.

STAGE 1 Completing the square

Ask yourself The integrand $y^2 \sec^2 x + 2yy' \tan x$ is an exact derivative. Of what?

STAGE 2 When is $I = 0$?

Ask yourself If the integral of a continuous non-negative function is zero, what can you say about the function? So what equation must y satisfy if $I = 0$?

STAGE 3 The general case

Ask yourself Repeat Stage 1 with $\int_0^1 (y' + ay \tan bx)^2 \, dx$. What must b be for the leftover terms to form an exact derivative?

Pause and try Where can this argument break down? (Look at $\tan ax$ on the interval $0 \leq x \leq 1$.)

STAGE 4 The critical case

Ask yourself When $a = \frac{\pi}{2}$, the proof fails. In Stage 2, the function making $I = 0$ would have satisfied $y' + y \tan x = 0$. What is the analogous equation now, and what are its solutions?

Try this next

1 **Beyond the critical value.** Show that if $a > \frac{\pi}{2}$ then $y = \cos \frac{\pi x}{2}$ gives $J < 0$.

2 **The Euler–Lagrange equation.** HARDERSuppose y minimises J among functions with $y(1) = 0$. For any function η with $\eta(1) = 0$ and any number ε, $J(y + \varepsilon\eta) \geq J(y)$. Expand $J(y + \varepsilon\eta)$ in powers of ε and deduce that $\int_0^1 (y'\eta' - a^2 y\eta) \, dx = 0$. Integrate by parts to show that $y'' + a^2 y = 0$ and $y'(0) = 0$ (assuming y is smooth enough).

3 **A longer interval.** Show by a substitution that if $y(L) = 0$, then $\int_0^L \left((y')^2 - a^2y^2\right)\,\mathrm{d}x \geq 0$ for all y exactly when $a < \frac{\pi}{2L}$.

4 **The cycloid.** Look up the brachistochrone problem and find the parametric equations of the cycloid. Show that a bead released from rest anywhere on an upside-down cycloid reaches the bottom in the same time (the *tautochrone* property, found by Huygens in 1659).

Further reading

- **Paul Nahin, *When Least Is Best*** (Princeton University Press). The history and mathematics of optimisation, including the brachistochrone, written for students.
- **Richard Feynman, *The Feynman Lectures on Physics*,** Volume II, Chapter 19 ("The principle of least action"). Free online.
- **Bruce van Brunt, *The Calculus of Variations*** (Springer). An undergraduate textbook; the first chapters need only A-level calculus.

The full guided solution, and what the question leads to at university, is in Section 4.7:
https://keithmcnulty.github.io/step_book_mcnulty/ch04/variations.html