

4.6 Integrals With No Formula

WORKSHEET

The question

(i) Given that

$$\int \frac{x^3 - 2}{(x+1)^2} e^x dx = \frac{P(x)}{Q(x)} e^x + \text{constant},$$

where $P(x)$ and $Q(x)$ are polynomials, show that $Q(x)$ has a factor of $x + 1$.

Show also that the degree of $P(x)$ is exactly one more than the degree of $Q(x)$, and find $P(x)$ in the case $Q(x) = x + 1$.

(ii) Show that there are no polynomials $P(x)$ and $Q(x)$ such that

$$\int \frac{1}{x+1} e^x dx = \frac{P(x)}{Q(x)} e^x + \text{constant}.$$

You need consider only the case when $P(x)$ and $Q(x)$ have no common factors.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

? **Ask yourself** If $\int f(x)e^x dx = \frac{P}{Q}e^x + C$, what equation must P and Q satisfy? (Differentiate the right-hand side.)

STAGE 1 The factor $x + 1$

? **Ask yourself** Put $x = -1$ in (†). What must happen?

STAGE 2 Degrees

? **Ask yourself** In $P'Q - PQ' + PQ$, which term has the highest degree? Can it be cancelled by the others?

STAGE 4 An impossibility

⌚ **Pause and try** Substitute $Q = (x + 1)^k R$, cancel as many factors of $(x + 1)$ as you can, and then put $x = -1$. What do you learn about $P(-1)$?

Try this next

- Two that work.** Show that $\int \frac{xe^x}{(x+1)^2} dx = \frac{e^x}{x+1} + C$ and $\int \frac{(x^2+1)e^x}{(x+1)^2} dx = \frac{x-1}{x+1}e^x + C$.
- The Gaussian.** Show that there is no polynomial P with $\int e^{-x^2} dx = P(x)e^{-x^2} + C$. (*Hint: differentiate and compare degrees.*)
- Another pole.** Adapt the argument of part (ii) to show that $\int \frac{e^x}{x} dx$ is not of the form $\frac{P(x)}{Q(x)}e^x + C$.
- When is it possible?** Suppose f is a polynomial and $\int \frac{f(x)}{(x+1)^2} e^x dx = \frac{P(x)}{x+1} e^x + C$ for some polynomial P . Show that $(x+1)P' + xP = f$, and by evaluating this and its derivative at $x = -1$, show that $f(-1) + f'(-1) = 0$. Check that this condition holds for $f(x) = x^3 - 2$ and for the examples in exercise 1. **HARDER** Can you show that the condition is also sufficient?

Further reading

- ▶ **Brian Conrad**, “**Impossibility theorems for elementary integration**”. Freely available lecture notes explaining Liouville’s theorem with examples.
- ▶ **Elena Anne Marchisotto & Gholam-Ali Zakeri**, “**An invitation to integration in finite terms**”, *The College Mathematics Journal* 25 (1994). A gentle introduction to Liouville’s theory.
- ▶ **Manuel Bronstein**, ***Symbolic Integration I*** (Springer). How computer algebra systems actually integrate. Advanced.

The full guided solution, and what the question leads to at university, is in Section 4.6:
https://keithmcnulty.github.io/step_book_mcnulty/ch04/no-formula.html