

4.5 The Sophomore's Dream

WORKSHEET

The question

Let m be a positive integer and let n be a non-negative integer.

- (i) Use the result $\lim_{t \rightarrow \infty} e^{-mt} t^n = 0$ to show that

$$\lim_{x \rightarrow 0} x^m (\ln x)^n = 0.$$

By writing x^x as $e^{x \ln x}$ show that

$$\lim_{x \rightarrow 0} x^x = 1.$$

- (ii) Let $I_n = \int_0^1 x^m (\ln x)^n dx$. Show that

$$I_{n+1} = -\frac{n+1}{m+1} I_n$$

and hence evaluate I_n .

- (iii) Show that

$$\int_0^1 x^x dx = 1 - \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^3 - \left(\frac{1}{4}\right)^4 + \dots.$$

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

- ?** **Ask yourself** The function x^x is not defined at $x = 0$ by any formula you know (0^0 is ambiguous). How does it behave as x approaches 0 from above? Try $x = 0.1, 0.01, 0.001$ on a calculator.

STAGE 1 Powers beat logarithms

- ?** **Ask yourself** As $x \rightarrow 0^+$, $\ln x \rightarrow -\infty$. What substitution turns $x \rightarrow 0^+$ into $t \rightarrow \infty$, and turns $x^m(\ln x)^n$ into something like $e^{-mt}t^n$?

STAGE 2 A reduction formula

- ?** **Ask yourself** Integrate $I_{n+1} = \int_0^1 x^n (\ln x)^{n+1} dx$ by parts. Which factor should you integrate? What happens to the boundary term at $x = 0$?

STAGE 3 The series

- ?** **Ask yourself** Expand $x^x = e^{x \ln x}$ using the series for e^y . What does each term look like? Which value of m makes the integrals in part (ii) appear?

- ✓** **Pause and try** Integrate term by term from 0 to 1, using part (ii) with $m = n$ (and treating $n = 0$ separately). What do you get?

Try this next

- 1 The companion.** Show that $\int_0^1 x^{-x} dx = \sum_{n=1}^{\infty} \frac{1}{n^n} \approx 1.2913$.
- 2 The minimum.** Show that x^x has its minimum on $(0, \infty)$ at $x = \frac{1}{e}$, with value $e^{-1/e}$.
- 3 How many terms?** For an alternating series with decreasing terms, the error after stopping is less than the first omitted term. How many terms of the series for $\int_0^1 x^x dx$ are needed for six decimal places?
- 4 A cousin.** Use the same method to show that $\int_0^1 x^2 \ln x dx = -\frac{1}{9}$, and that

$$\int_0^1 \frac{\ln x}{1-x} dx = -\sum_{n=1}^{\infty} \frac{1}{n^2}. \text{ (By Section 2.4, this is } -\frac{\pi^2}{6}.)$$

Further reading

- **Jonathan Borwein, David Bailey & Roland Girgensohn, *Experimentation in Mathematics*** (A.K. Peters). The source of the name, and many more identities discovered by computer.
- **William Dunham, *The Calculus Gallery*** (Princeton University Press). Masterpieces of analysis from Newton to Lebesgue, including Bernoulli's work.
- **Mary Hart, *Guide to Analysis*** (Palgrave). Clear treatment of uniform convergence and when sums and integrals can be swapped.

The full guided solution, and what the question leads to at university, is in Section 4.5:
https://keithmcnulty.github.io/step_book_mcnulty/ch04/sophomores-dream.html