

4.4 Wallis, Beta and the Factorial of a Half

WORKSHEET

The question

Let

$$u_n = \int_0^{\frac{1}{2}\pi} \sin^n t \, dt$$

for each integer $n \geq 0$. By integrating

$$\int_0^{\frac{1}{2}\pi} \sin t \sin^{n-1} t \, dt$$

by parts, or otherwise, obtain a formula connecting u_n and u_{n-2} when $n \geq 2$ and deduce that

$$nu_n u_{n-1} = (n-1)u_{n-1}u_{n-2}$$

for all $n \geq 2$. Deduce that

$$nu_n u_{n-1} = \frac{1}{2}\pi.$$

Sketch graphs of $\sin^n t$ and $\sin^{n-1} t$, for $0 \leq t \leq \frac{1}{2}\pi$, on the same diagram and explain why $0 < u_n < u_{n-1}$. By using the result of the previous paragraph show that

$$nu_n^2 < \frac{1}{2}\pi < nu_{n-1}^2$$

for all $n \geq 1$. Hence show that

$$\left(\frac{n}{n+1}\right) \frac{1}{2}\pi < nu_n^2 < \frac{1}{2}\pi$$

and deduce that $nu_n^2 \rightarrow \frac{1}{2}\pi$ as $n \rightarrow \infty$.

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.



Ask yourself What are u_0 and u_1 ? And roughly what does the graph of $\sin^n t$ look like on $[0, \frac{\pi}{2}]$ when n is large?

STAGE 1 A reduction formula



Ask yourself Integrate $\int \sin t \cdot \sin^{n-1} t \, dt$ by parts, integrating $\sin t$ and differentiating $\sin^{n-1} t$. The new integrand contains $\cos^2 t$; how can you get back to powers of $\sin t$?

STAGE 2 An invariant



Ask yourself The relation says the quantity $nu_n u_{n-1}$ does not change when n goes down by one. What is it equal to at $n = 1$?

STAGE 3 A squeeze



Pause and try Use $u_n < u_{n-1}$ and $nu_n u_{n-1} = \frac{1}{2}\pi$ to trap nu_n^2 .

Try this next

- Wallis's product.** Using the explicit values of u_{2m} and u_{2m+1} , and $u_{2m+1} < u_{2m} < u_{2m-1}$, prove Wallis's product.
- A Beta integral.** (From STEP II, 2013.) Let $I_n = \int_0^1 x^n (1-x)^n \, dx$. Show by integration by parts and symmetry that $I_n = \frac{n}{2(2n+1)} I_{n-1}$, and deduce that $I_n = \frac{(n!)^2}{(2n+1)!}$. Use $x = \sin^2 \theta$ to show that $I_{1/2} = \frac{\pi}{8}$. What does the formula $I_n = \frac{(n!)^2}{(2n+1)!}$ suggest $\left(\frac{1}{2}\right)!$ should be?
- The Beta function.** (From STEP III, 2025.) For $p, q > 0$ let $b(p, q) = \int_0^1 x^{p-1} (1-x)^{q-1} \, dx$. Show that $b(p, q) = b(q, p)$, that $b(p+1, q) = b(p, q) - b(p, q+1)$, and, with the substitution $x = \frac{t}{1+t}$, that $b(p, q) = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}} \, dt$. Hence evaluate $\int_0^\infty \frac{t^{3/2}}{(1+t)^6} \, dt$. (The answer is $\frac{3\pi}{256}$.)
- Central binomial coefficients.** **HARDER** Show that $u_{2m} = \binom{2m}{m} \frac{\pi}{2^{2m+1}}$, and use $nu_n^2 \rightarrow \frac{\pi}{2}$

to deduce that $\binom{2m}{m} \approx \frac{4^m}{\sqrt{\pi m}}$ for large m . Check this for $m = 10$.

Further reading

- **Julian Havil, *Gamma*** (Princeton University Press). Includes the Gamma function and its history.
- **Philip J. Davis, “Leonhard Euler’s integral: a historical profile of the Gamma function”**, *American Mathematical Monthly* 66 (1959). A classic, prize-winning article, readable with this section behind you.
- **Emil Artin, *The Gamma Function*** (Dover). A short, beautiful monograph for when you have done some analysis.

The full guided solution, and what the question leads to at university, is in Section 4.4:
https://keithmcnulty.github.io/step_book_mcnulty/ch04/wallis-beta.html