

4.3 $22/7$ Is Bigger Than π

WORKSHEET

The question

Find constants $a_0, a_1, a_2, a_3, a_4, a_5, a_6$ and b such that

$$x^4(1-x)^4 = (a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0)(x^2 + 1) + b.$$

Hence, or otherwise, prove that

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx = \frac{22}{7} - \pi.$$

Evaluate $\int_0^1 x^4(1-x)^4 dx$ and deduce that

$$\frac{22}{7} > \pi > \frac{22}{7} - \frac{1}{630}.$$

Hints

Try the question on your own first. If you get stuck, read the hints *one at a time*, and go back to the question after each one. Hints marked *Pause and try* are the key steps.

? **Ask yourself** Why does π come out of $\int_0^1 \frac{\text{polynomial}}{1+x^2} dx$? Which simple integral of this form gives π on its own?

STAGE 1 Polynomial division

? **Ask yourself** Divide by $x^2 + 1$, either by long division or by comparing coefficients. What is the remainder?

STAGE 3 Bounding the integral

? **Ask yourself** The integrand is positive on $(0, 1)$, so $\frac{22}{7} > \pi$ at once. To get a lower bound for π we need an *upper* bound for the integral. How big can $\frac{1}{1+x^2}$ be on $[0, 1]$?

✓ **Pause and try** Evaluate $\int_0^1 x^4(1-x)^4 dx$, and finish the proof.

Try this next

1 Check the numbers. Compute $\frac{22}{7} - \pi$ to four significant figures and confirm that it lies between $\frac{1}{1260}$ and $\frac{1}{630}$.

2 A Beta integral. Use the formula $\int_0^1 x^n(1-x)^n dx = \frac{(n!)^2}{(2n+1)!}$ (proved in Section 4.4) to check that $\int_0^1 x^4(1-x)^4 dx = \frac{1}{630}$.

3 A lower approximation. Show that

$$\int_0^1 \frac{x(1-x)^2}{1+x^2} dx = \frac{\pi}{2} - \frac{3}{2} \quad \text{and} \quad \int_0^1 \frac{x^2(1-x)^4}{1+x^2} dx = \pi - \frac{47}{15},$$

and deduce that $\pi > 3$ and $\pi > \frac{47}{15}$. Why are these less accurate than $\frac{22}{7}$?

4 Going further. **HARDER** Verify the identity for $\int_0^1 \frac{x^8(1-x)^8}{1+x^2} dx$ given above. (A computer algebra system will do the division; the point is to see that the remainder is 16, giving 4π .)

Further reading

- ▶ **Petr Beckmann, *A History of π*** (St Martin's Press). Archimedes' polygon method in detail, and much else.
- ▶ **Stephen Lucas, "Approximations to π derived from integrals with nonnegative integrands"**, *American Mathematical Monthly* 116 (2009). Short and readable.
- ▶ **Ivan Niven, *Irrational Numbers*** (Mathematical Association of America). Includes Niven's proof that π is irrational.

The full guided solution, and what the question leads to at university, is in Section 4.3:
https://keithmcnulty.github.io/step_book_mcnulty/ch04/twenty-two-sevenths.html